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PREP BOOK

# MATHEMATICS

## FUNG

- Chapter-wise Complete Syllabus
- Comprehensive Course Revision
- Detailed Explanation of Topics
- Key Points, Tables, Flow Sheet & Graphs
- Easy to Remember; Important Key Facts

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# BOOK-I

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# 1 CHAPTER NUMBER SYSTEMS

## Table of Sets

| NAME | DEFINITION |
|---|---|
| Set of Natural Numbers | $N = \{1, 2, 3, \cdots\}$ |
| Set of Whole Numbers | $W = \{0, 1, 2, 3, \cdots\}$ |
| Set of Integers | $Z = \{0, \pm 1, \pm 2, \pm 3, \cdots\}$ |
| Set of Positive Integers | $Z^+ = \{1, 2, 3, \cdots\} = N$ |
| Set of Negative Integers | $Z^- = \{-1, -2, -3, \cdots\}$ |
| Set of Non-negative Integers | $\{0, 1, 2, 3, \cdots\} = W$ |
| Set of Non-positive Integers | $\{0, -1, -2, -3, \cdots\}$ |
| Set of Odd Integers | $O = \{\pm 1, \pm 3, \pm 5, \cdots\}$ |
| Set of Even Integers | $E = \{0, \pm 2, \pm 4, \cdots\}$ |
| Set of Prime Numbers | $P = \{2, 3, 5, 7, 11, 13, 17, 19, \cdots\}$ |
| Set of Rational Numbers | $Q = \left\{x \mid x = \frac{p}{q}, p, q \in Z, q \neq 0\right\}$ |
| Set of Irrational Numbers | $Q' = \left\{x \mid x \neq \frac{p}{q}, p, q \in Z, q \neq 0\right\}$ |
| Set of Real Numbers | $\mathbb{R} = Q \cup Q'$ |

$Q \cap Q' = \phi$



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## Decimal Representation of Rational and Irrational Numbers

| NAME | DEFINITION | EXAMPLES |
|---|---|---|
| Terminating Decimals | A decimal having finite number of digits in its decimal part. | 434.77779 , 684.765003 0.000000000001 |
| Non Terminating Decimals | A decimal having infinite number of digits in its decimal part. | 3.1452… 76.9000123… |
| Recurring Decimals Or Periodic Decimals | A decimal in which one or more digits repeat indefinitely in its decimal part. | $3.13131313\cdots = 3.\overline{13}$ $1.121121121\cdots = 1.\overline{121}$ |
| Irrational Numbers | All non terminating and non recurring decimals represents Irrational numbers. | 2.31415162… $3.141592654\cdots = \pi$ $\sqrt{n}$ ,where $n$ is prime |
| Rational Numbers | All Terminating decimals and Recurring decimals represents Rational numbers. | $0.78$ $4.\overline{424} = 4.424424...$ |

**Note:**

(i) $\pi = \dfrac{\text{Circumference of any circle}}{\text{Length of its diameter}}$

(ii) $\dfrac{22}{7}$ and $\dfrac{355}{113}$ are the approximate values of $\pi$

## Operations

| NAME OF OPERATION | DEFINITION | EXAMPLES |
|---|---|---|
| Unary Operations | An operation which converts one number into other. | Negation of number, squaring a number, extracting square root, trigonometric operations, differentiation, integration etc |
| Binary Operations | (i) An operation which converts two numbers into third.<br>(ii) A rule denoted by * in a set $S$ that assigns to any pair of elements of $S$ taken in a definite order to another element of $S$ .<br>(iii) A function from $A \times A$ to the set $A$ . | Following are some binary operations.<br>Addition(+),<br>multiplication(×),<br>division(÷),<br>subtraction( − ),<br>union, intersection and difference of sets.<br>The most important binary operations are + and × |



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# REAL NUMBER SYSTEM

## Properties of Real Numbers

| PROPERTY/LAW | ADDITION | MULTIPLICATION |
|---|---|---|
| Closure Law | $\forall x, y \in \mathbb{R}, x+y \in \mathbb{R}$ | $\forall x, y \in \mathbb{R}, x.y \in \mathbb{R}$ |
| Associative Law | $\forall x, y, z \in \mathbb{R}, x+(y+z)=(x+y)+z$ | $\forall x, y, z \in \mathbb{R}$, $x.(y.z)=(x.y).z$ |
| Identity Property | $\forall x \in \mathbb{R}, \exists 0 \in \mathbb{R}$ such that $x+0=0+x=x$, 0 is identity element | $\forall x \in \mathbb{R}$ , $\exists 1 \in \mathbb{R}$ such that $x.1=1.x=x$, 1 is identity element |
| Inverse Property | $\forall x \in \mathbb{R}, \exists(-x) \in \mathbb{R}$ such that $x+(-x)=(-x)+x=0$ | $\forall x \in \mathbb{R}(x \neq 0), \exists \frac{1}{x} \in \mathbb{R}$ such that $x.\frac{1}{x}=\frac{1}{x}.x=1$ |
| Commutative Law | $\forall x, y \in \mathbb{R}$ , $x+y=y+x$ | $\forall x, y \in \mathbb{R}$ , $x.y=y.x$ |
| Distributive Law | (**Distributivity** of **Multiplication** over **Addition**) $\forall x, y, z \in \mathbb{R}$ (i) $x(y+z)=xy+xz$ (Left distributive law) (ii) $(x+y)z=xz+yz$ (Right distributive law) | |

## Field

Any set possessing all above 11 properties is called a field. Hence the set of real numbers is a field.

## Properties of Equality of Real Numbers

| NAME OF PROPERTY | DEFINITION |
|---|---|
| Reflexive Property | $\forall x \in \mathbb{R}, x=x$ |
| Symmetric Property | $\forall x, y \in \mathbb{R}, x=y \Rightarrow y=x$ |
| Transitive Property | $\forall x, y, z \in \mathbb{R}, x=y \wedge y=z \Rightarrow x=z$ |
| Additive Property | $\forall x, y, z \in \mathbb{R}, x=y \Rightarrow x+z=y+z$ |
| Multiplicative Property | $\forall x, y, z \in \mathbb{R}, x=y \Rightarrow x.z=y.z$ |
| Cancellation Property w.r.t. Addition | $\forall x, y, z \in \mathbb{R}, x+z=y+z \Rightarrow x=y$ |
| Cancellation Property w.r.t. Multiplication | $\forall x, y, z \in \mathbb{R}, x.z=y.z \Rightarrow x=y, z \neq 0$ |

## Properties of Inequalities of Real Numbers (Order Properties)

| NAME OF THE PROPERTY | DEFINITION |
|---|---|
| Trichotomy Property | $\forall a, b \in \mathbb{R}$ either $a=b$ or $a<b$ or $a>b$ |
| Transitive Property | $\forall a, b, c \in \mathbb{R}$ (i) $a>b \wedge b>c \Rightarrow a>c$ (ii) $a<b \wedge b<c \Rightarrow a<c$ |



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| | |
|---|---|
| Additive Property | $\forall a,b,c,d \in \mathbb{R}$<br>**(a)**<br>(i) $a>b \Rightarrow a+c>b+c$<br>(ii) $a<b \Rightarrow a+c<b+c$<br>**(b)**<br>(i) $a>b \wedge c>d \Rightarrow a+c>b+d$<br>(ii) $a<b \wedge c<d \Rightarrow a+c<b+d$ |
| Multiplicative Property | $\forall a,b,c \in \mathbb{R}$<br>**(a)**<br>(i) $a>b \Rightarrow ac>bc\,(c>0)$<br>(ii) $a<b \Rightarrow ac<bc\,(c>0)$<br>**(b)**<br>(i) $a>b \Rightarrow ac<bc\,(c<0)$<br>(ii) $a<b \Rightarrow ac>bc\,(c<0)$<br>**(c)**<br>$\forall a,b,c,d \in \mathbb{R}$ and $a,b,c,d$ are all positive<br>(i) $a>b \wedge c>d \Rightarrow ac>bd$<br>(ii) $a<b \wedge c<d \Rightarrow ac<bd$ |

## Properties of Fractions of Real Numbers

| NAME OF THE PROPERTY | DEFINITION |
|---|---|
| Principle for Equality of Fractions | $\frac{a}{b}=\frac{c}{d} \Leftrightarrow ad=bc$ |
| Rule for Product of Fractions | $\frac{a}{b}.\frac{c}{d}=\frac{ac}{bd}$ |
| Golden Rule of Fractions | $\frac{a}{b}=\frac{ka}{kb},(k\neq 0)$ |
| Rule for Quotient of Fractions | $\frac{\frac{a}{b}}{\frac{c}{d}}=\frac{ad}{bc}$ |

## Complex Numbers

**Definition:**

The numbers of the form $x+iy$ where $x,y \in \mathbb{R}$ and $i=\sqrt{-1}$ are called complex numbers. The set of complex numbers is denoted by $\mathbb{C}$.

- In $z=x+iy$, $x$ is called **Real Part** and $y$ is called **Imaginary Part.**
- $x+iy$ can be written in **Ordered Pair** form as $(x,y)$.
- The product of real number and $i$ is called an **Imaginary Number** or **Pure Imaginary Number**. e.g. $i,-3i,\sqrt{5}i$ etc.

**0 is both pure real and pure imaginary**



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## Integral Powers of *i*:

- $i = \sqrt{-1}$
- $i^2 = -1$
- $i^3 = -i$
- $i^4 = 1$
- $i^{4n} = 1$ where $n \in \mathbb{Z}$
- $i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0$ , $n \in \mathbb{Z}$

## Operations on Complex Numbers

| NAME OF OPERATION | DEFINITION |
|---|---|
| Equality of complex numbers | $\left.\begin{matrix} a+ib=c+id \\ (a,b)=(c,d) \end{matrix}\right\} \Leftrightarrow a=c \wedge b=d$ |
| Addition of complex numbers | $a+ib+c+id=(a+c)+i(b+d)$ or $(a,b)+(c,d)=(a+c,b+d)$ |
| Scalar multiplication of complex numbers | $k(a+ib)=ka+ikb$ $k(a,b)=(ka,kb)$ |
| Multiplication of complex numbers | $(a+ib)(c+id)=(ac-bd)+(ad+bc)i$ $(a,b)(c,d)=(ac-bd,ad+bc)$ |
| Division of complex numbers | $\frac{(a,b)}{(c,d)}=\left(\frac{ac+bd}{c^2+d^2},\frac{bc-ad}{c^2+d^2}\right)$ |

## Properties of Fundamental Operations on Complex Numbers

| PROPERTY | ADDITION | MULTIPLICATION |
|---|---|---|
| Identity property | $(0,0)=0+0.i$ | $(1,0)=1+0.i$ |
| Inverse property | Each $(a,b) \in \mathbb{C}$ has additive inverse $(-a,-b)$ $i.e.\ (a,b)+(-a,-b)=(0,0)$ | Every non zero complex number $(a,b)$ has an inverse $\left(\frac{a}{a^2+b^2},\frac{-b}{a^2+b^2}\right)$ $i.e.\ (a,b)\left(\frac{a}{a^2+b^2},\frac{-b}{a^2+b^2}\right)$ $=\left(\frac{a}{a^2+b^2},\frac{-b}{a^2+b^2}\right)(a,b)=(1,0)$ |
| Distributive property | Multiplication is distributive over addition in $\mathbb{C}$ i.e. $(a,b)\left[(c,d)\pm(e,f)\right]=(a,b)(c,d)\pm(a,b)(e,f)$ | |



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## Conjugate of a Complex Number

Let $z = x + iy = (x, y)$ be a complex number, then the conjugate of z is denoted by $\overline{z}$ and is defined as $\overline{z} = x - iy = (x, -y)$.

**Complex numbers do not hold order axioms.**

## Properties of Conjugate of Complex Number

$\forall z, z_1, z_2 \in \mathbb{C}$

**A real number is self conjugate**

(i) $\overline{(\overline{z})} = z$

(ii) $z\overline{z} = |z|^2$

(iii) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ , $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$

(iv) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}, (z_2 \neq 0)$, $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$

(v) $\frac{1}{z} = \frac{\overline{z}}{|z|^2}, (z \neq 0)$

(vi) $z + \overline{z} = 2\mathrm{Re}(z)$

(vii) $z - \overline{z} = 2i\mathrm{Im}(z)$

(viii) $z = \overline{z}$ iff $z$ is real

(ix) $z = -\overline{z}$ iff $z$ is pure imaginary

## Modulus of a Complex Number

Let $z = x + iy = (x, y)$ then

Modulus of $z = |z| = \sqrt{x^2 + y^2}$

**The modulus of a complex number is the distance from the origin to the point representing the number on the graph.**

## Properties of Modulus of a Complex Number

$\forall z, z_1, z_2 \in \mathbb{C}$

**Note that $|z| \geq 0$**

(i) $|z| = |-z| = |\overline{z}| = |-\overline{z}|$

(ii) $|z_1 . z_2| = |z_1| . |z_2|$

(iii) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ for $z_2 \neq 0$

(iv) $\big||z_1| - |z_2|\big| \leq |z_1 + z_2| \leq |z_1| + |z_2|$

(v) $|z_1 + z_2| \leq |z_1| + |z_2|$ is analogue of the triangular inequality.

(vi) $|z^n| = |z|^n , \forall n \in Z$



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## Graphical Representation of Complex Numbers

Since a complex number is considered as an order pair of real numbers, so we can represent such numbers by points in a xy-plane (Cartesian plane) which is called **complex plane** or **z-plane.** In this representation $x-axis$ is called **Real Axis** and $y-axis$ is called **Imaginary Axis**. The figure representing one or more complex numbers on the complex plane is called an **Argand diagram.**

## Argument of Complex Number

Let $z = x + iy = (x, y)$ then

Argument of z = $\arg z = \tan^{-1}\left(\frac{y}{x}\right)$

## Principal Argument of a Complex Number

The principal argument $\theta$ of a complex number $x+iy$ is defined in the range $-\pi < \theta \leq \pi$, such that

- In 1st Quadrant, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$
- In 2nd Quadrant, $\theta = \pi - \tan^{-1}\left(\frac{y}{|x|}\right)$
- In 3rd Quadrant, $\theta = \tan^{-1}\left(\frac{|y|}{|x|}\right) - \pi$
- In 4th Quadrant, $\theta = -\tan^{-1}\left(\frac{|y|}{x}\right)$

## Properties of Argument of Complex Number

(i) $\arg(z_1.z_2) = \arg(z_1) + \arg(z_2)$

(ii) $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

(iii) $\arg(\overline{z}) = -\arg(z) = \arg\left(\frac{1}{z}\right)$

(iv) $\arg(z.\overline{z}) = 0$

(v) $\arg\left(\frac{z}{\overline{z}}\right) = 2\arg z$

(vi) $\arg(z^n) = n\arg(z),\ n \in \mathbb{Z}$

**Argument of (0, 0) is not defined.**

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## Polar Form of a Complex Number

- Since every complex number $x+iy$ can be represented in the form of a point $(x,y)$ ,so we can express every complex number in the form of polar coordinates $(r,\theta)$, where $r=\sqrt{x^2+y^2}$ and $\theta=\tan^{-1}\left(\frac{y}{x}\right)$.
- $x+iy=r\cos\theta+ir\sin\theta$ or
- $x+iy=r(\cos\theta+i\sin\theta)$ or
- $x+iy=rcis\theta$
- To convert polar coordinates into Cartesian coordinates, replace $(r,\theta)$ by $(r\cos\theta,r\sin\theta)$

## De Moivre's Theorem

Let $z=x+iy=r(\cos\theta+i\sin\theta)$

Then $z^n=(x+iy)^n=r^n(\cos\theta+i\sin\theta)^n$

By **De Moivre's theorem**

$z^n=(x+iy)^n=r^n(\cos n\theta+i\sin n\theta),\forall n\in Z$

**Results:**

**(i)** $z_1.z_2=r_1.r_2(\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2))$ or $z_1.z_2=r_1.r_2cis(\theta_1+\theta_2)$

**(ii)** $\frac{z_1}{z_2}=\frac{r_1}{r_2}(\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2))$ or $\frac{z_1}{z_2}=\frac{r_1}{r_2}cis(\theta_1-\theta_2)$

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# 2 CHAPTER SETS, FUNCTIONS AND GROUPS

## Set*

A well-defined collection of distinct objects is called set.

e.g. $\{a,b,c,d\}$ and $\{1,2,3\}$

## Ways of Describing a Set

There are three ways to describe a set.

| WAY OF DESCRIPTION | DEFINITION | EXAMPLE |
|---|---|---|
| The Descriptive Method | In this method the set can be described in words. | The set of vowels in English alphabets |
| The Tabular Method | In this method the set is described by listing its elements with in brackets. | $A=\{a,e,i,o,u\}$ |
| Set Builder Method | In this method a symbol or letter is used for an arbitrary member of the set and stating the property common to all the members of the set. | $A=\{x \mid x \text{ is vowel}\}$ |

## Order of a Set

The number of elements present in a set is called order of the set.

## Types of Sets (w.r.t. the order)

| NAME OF THE SET | DEFINITION | EXAMPLE |
|---|---|---|
| Finite Set | If a set has definite number of elements present in it. | $\{1,2,3,\cdots,10\}$ |
| Infinite Set | If a set has indefinite number of elements present in it. | $N=\{1,2,3,\cdots\}$<br>$Z_{\circ}=\{0,\pm1,\pm2,\cdots\}$ |
| Null Set or Empty Set or Void Set | A set having no elements. It is denoted by $\phi$. | $\phi=\{\ \}$ |
| Singleton Set | A set having only one element. | $\{7\}$ |

*The theory of sets is attributed to the German Mathematics George Cantor (1845-1918).

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## Terms Regarding Sets

| NAME OF THE SET | DEFINITION | EXAMPLE |
|---|---|---|
| **Equal sets** | Two sets $A$ & $B$ are equal if every element of $A$ is in $B$ and vice versa. | If $A=\{c,a,t\}$ & $B=\{a,c,t\}$ then $A=B$ |
| **Equivalent or similar sets** | Two sets $A$ & $B$ are equivalent iff a one to one correspondence exists between the elements of the sets. It is denoted by $A\sim B$ or $A\equiv B$ | If $A=\{1,2,3,4,5\}$ & $B=\{a,e,i,o,u\}$ then $A\sim B$ similarly $N\sim W$ where $N$ and $W$ are set of natural numbers and whole numbers respectively |
| **Subset** | If every element of a set $A$ is an element of a set $B$ ,then $A$ is a subset of $B$. written as $A\subseteq B$ | Let $A=\{1,2,3\}$ & $B=\{1,2,3,\cdots,10\}$ then $A\subseteq B$ |
| **Proper Subset** | If $A\subseteq B$ and $B$ contains at least one element which is not an element of the set $A$ then $A$ is called proper subset of $B$ and it is denoted by $A\subset B$ | In the above example $A$ is proper subset of $B$. |
| **Improper subset** | An improper subset is a subset which can be equal to the original set. | If $A=\{a,b,c\}$ so $A\subseteq A$ and $A=A$, so $A$ is improper subset of $A$ |
| **Superset** | If $A$ is proper subset of $B$ represented by $A\subset B$ then $B$ is called superset of $A$. It is denoted by $B\supset A$. | Let $A=\{1,2,3\}$ & $B=\{1,2,3,\cdots,10\}$ then $B\supset A$ |
| **Power set** | The collection of all proper and improper subsets of a set A is called power set of A. denoted by $P(A)$ | If $A=\{1,2\}$ then $P(A)=\{\phi,\{1\},\{2\},\{1,2\}\}$ |
| **Universal set or Universe of Discourse** | The super set of all the sets under a particular discussion. It is denoted by $U,E$ or $X$ | We can take set of Real numbers $R$ as universal set when we discuss set of even numbers and set of odd numbers. |

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## Operations on Sets

| NAME OF OPERATION | DEFINITION |
|---|---|
| Union of Two Sets | $A \cup B = \{x \mid x \in A \vee x \in B\}$ |
| Intersection of Two Sets | $A \cap B = \{x \mid x \in A \wedge x \in B\}$ |
| Difference of Two Sets | $A - B = \{x \mid x \in A \wedge x \notin B\}$ |

**Key Facts:**

- If a set has $n$-elements then it has $2^n$ subsets.
- If a set has $n$-elements then there are $2^n$ elements in its power set.

## Sets, Based on the Operations

| NAME OF THE SET | DEFINITION |
|---|---|
| Disjoint Sets | Two sets $A$ & $B$ are disjoint iff $A \cap B = \phi$. |
| Overlapping Sets | Two sets $A$ & $B$ are overlapping if $A \cap B \neq \phi$ but neither $A \subseteq B$ nor $B \subseteq A$ |
| Complement of a Set | $A' = A^c = U - A = \{x / x \in U \wedge x \notin A\}$ where $U$ is universal set and $A \subseteq U$ |

**Do you know:**

- The empty set has no proper subsets.
- An empty set is a proper subset of every non empty set.
- Every set is an improper subset of itself.

## Properties of Sets

| PROPERTY | DESCRIPTION |
|---|---|
| Commutative Property | $A \cup B = B \cup A,\ A \cap B = B \cap A$ |
| Associative Property | $A \cup (B \cup C) = (A \cup B) \cup C$<br>$A \cap (B \cap C) = (A \cap B) \cap C$ |
| Distributive Property of Union over Intersection | $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ |
| Distributive Property of Intersection over Union | $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ |

## Laws on Sets

| LAW | DESCRIPTION |
|---|---|
| De Morgan's Laws | $(A \cup B)' = A' \cap B'$<br>$(A \cap B)' = A' \cup B'$ |
| The absorption Laws | $A \cap (A \cup B) = A$<br>$A \cup (A \cap B) = A$ |
| Idempotent Laws | $A \cup A = A$<br>$A \cap A = A$ |
| The Complement Laws | $A \cap A' = \phi$<br>$A \cup A' = U$ |
| Identity Laws | $A \cup \phi = A$<br>$A \cap U = U$ |
| Set Difference Law | $A - B = A \cap B'$ |
| Universal bound Laws | $A \cup U = U$<br>$A \cap \phi = \phi$ |

**Remember that**
In general
$A - B \neq B - A$
$A \cup B \neq A \cap B$



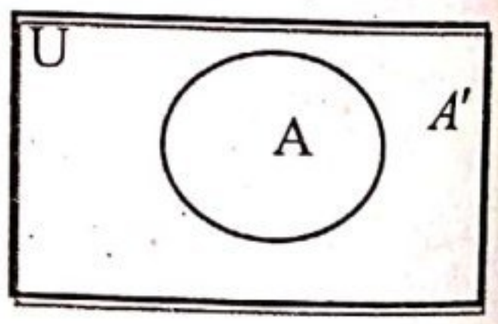
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## Theorems

- For any two sets $A$ & $B$ $A \subseteq (A \cup B)$ and $B \subseteq (A \cup B)$
- For any two sets $A$ & $B$ $(A \cap B) \subseteq A$ and $(A \cap B) \subseteq B$
- Let $U$ be the universal set then $U' = \phi$, $\phi' = U$
- For any two sets $A$ & $B$, $A \subset B \Rightarrow A \cap B = A$
- For any two sets $A$ & $B$, $A \subset B \Rightarrow A \cup B = B$
- For any two sets $A$ & $B$, $A \subseteq B \Rightarrow B' \subset A'$
- For any two sets $A$ & $B$, $A \subset B \Rightarrow A \cup (B - A) = B$
- For any two sets $A$ & $B$, $A \cap B' = A \Leftrightarrow A \cap B = \phi$
- For any three sets $A$,$B$ & $C$, $A - (B \cup C) = (A - B) \cap (A - C)$

## Venn Diagrams*

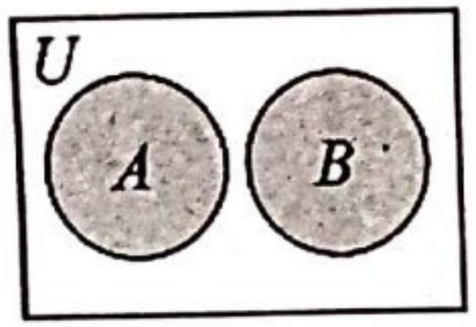
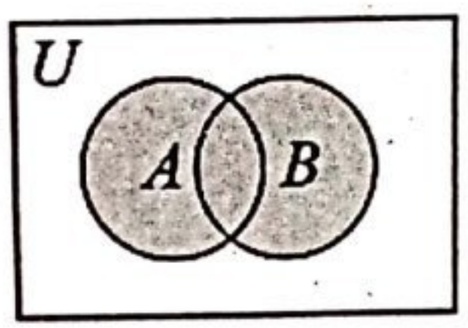
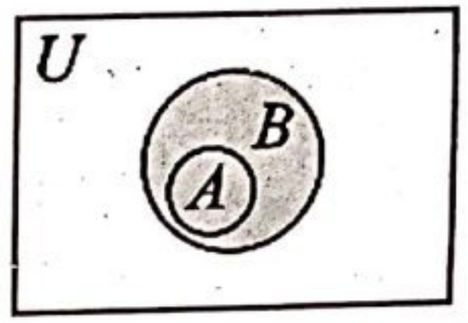
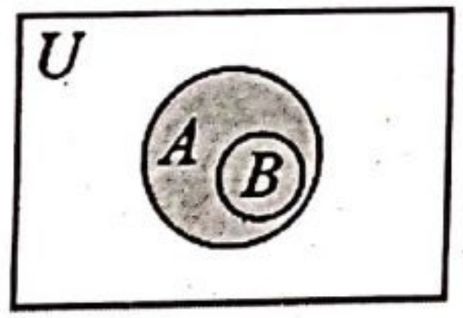
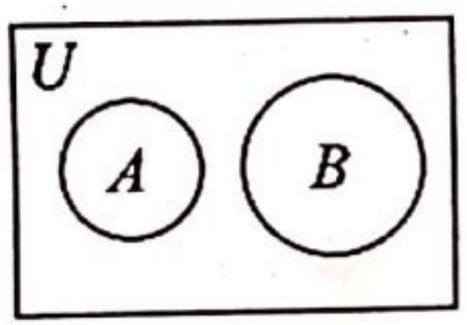
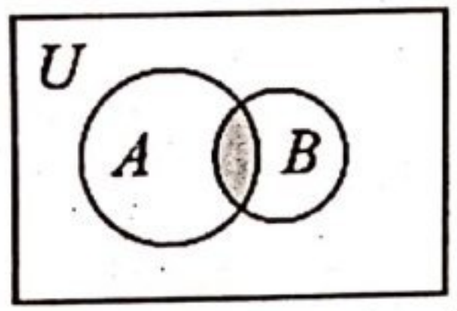
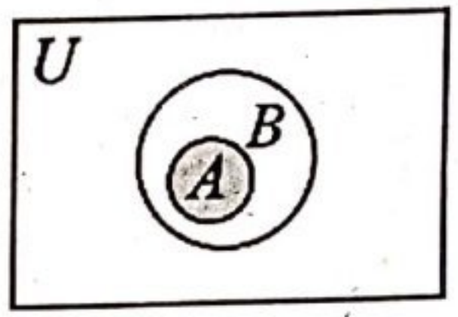
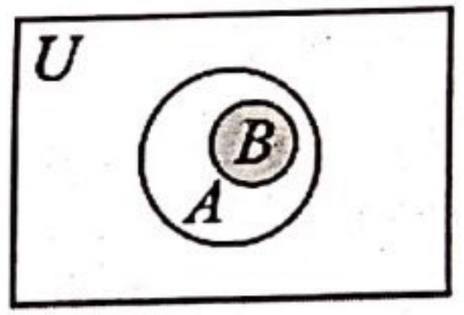
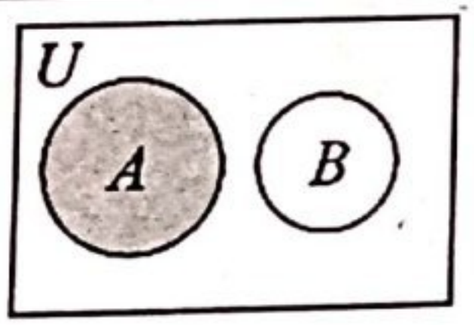
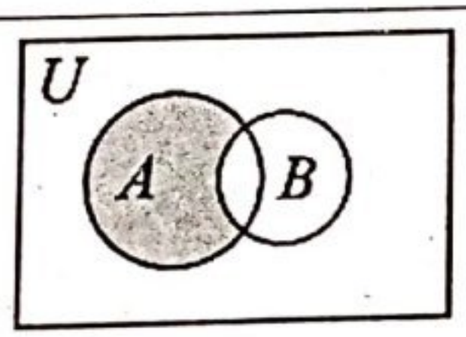
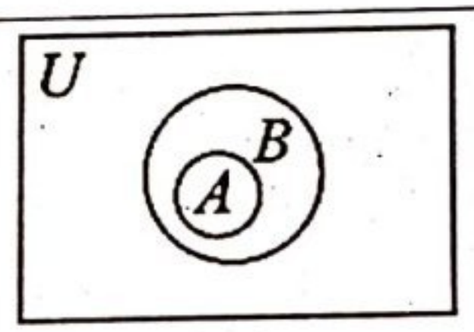
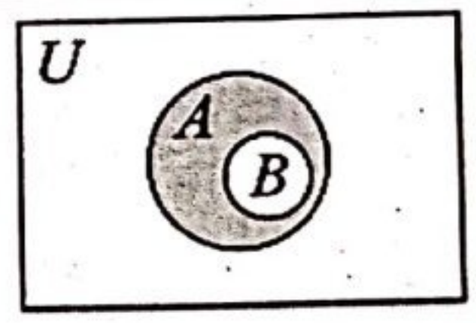
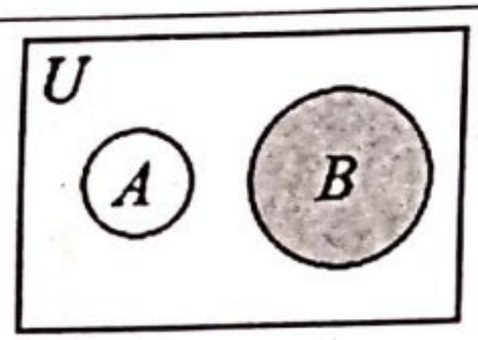
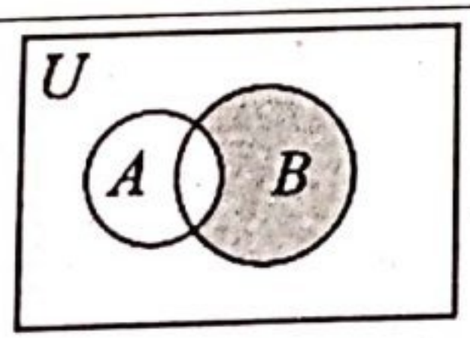
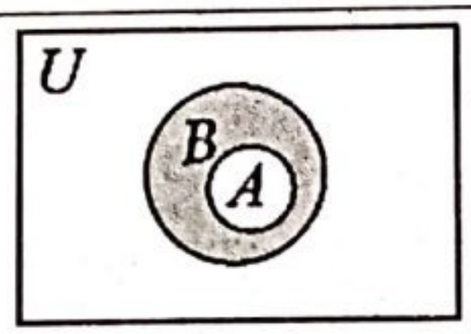
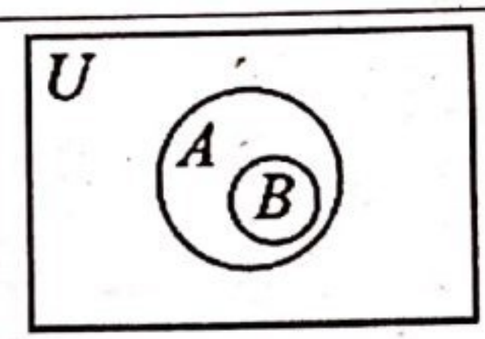
The pictorial representation of the sets is known as venn diagrams. In the venn diagram, the universal set $U$ is represented by the rectangular region and its subsets are represented by closed bounded region (preferred as circles) inside the rectangular region.




*Named after an English Mathematician John Venn (1834-1883).



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| | $A \cap B = \phi$ | $A \cap B \neq \phi$ | $A \subseteq B$ | $B \subseteq A$ |
|---|---|---|---|---|
| $A \cup B$ |   $n(A \cup B) = n(A) + n(B)$ |   $n(A \cup B) = n(A) + n(B) - n(A \cap B)$ |   $n(A \cup B) = n(B)$ |   $n(A \cup B) = n(A)$ |
| $A \cap B$ |   $n(A \cap B) = 0$ |   $n(A \cap B) = n(A) + n(B) - n(A \cup B)$ |   $n(A \cap B) = n(A)$ |   $n(A \cap B) = n(B)$ |
| $A - B$ |   $n(A - B) = n(A)$ |   $n(A - B) = n(A) - n(A \cap B)$ |   $n(A - B) = 0$ |   $n(A - B) = n(A) - n(B)$ |
| $B - A$ |   $n(B - A) = n(B)$ |   $n(B - A) = n(B) - n(A \cap B)$ |   $n(B - A) = n(B) - n(A)$ |   $n(B - A) = 0$ |



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## Logic[1]

| | |
|---|---|
| **Introduction** | The word logical means correctly reasoned[2]. |
| **Inductive Logic** | In inductive logic we can draw conclusions from limited number of observations or experiences. |
| **Deductive Logic** | In deductive logic the conclusions are drawn from the premises. |
| **Aristotelian Logic** | The logic based on the propositions. i.e. Only true or false possibility of a statement. |
| **Non-Aristotelian Logic** | A logic in which there are more chances other than true or false. |

## Proposition

A statement which is either true or false but not both is called a proposition.
*e.g.* Birds fly in the air, the sun rises in the east, $4 < 5$ are propositions.
Whereas open the door, go to Lahore etc are not propositions.

## Truth value

A truth value, some times called a logical value, is a value indicating the relation of a proposition to truth.

## Truth Table

A table consists of all the truth values of conditional is called truth table.
The following are the major statements in logic.

### 1. Negation ($\sim$ *or* $\neg$)

If $p$ is any proposition then negation of $p$ is denoted by $\sim p$ or $\neg p$ (read as not $p$) and is defined as if $p$ is true then $\neg p$ is false and vice versa.

Truth Table

| p | $\neg$p |
|---|---|
| T | F |
| F | T |

### 2. Conjunction ($\wedge$)

The conjunction $p \wedge q$ (read as p and q) is true when both p and q are true otherwise always false.

Truth Table

| p | q | $p \wedge q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

### 3. Disjunction ($\vee$)

The disjunction $p \vee q$ (read as p or q) is false when both p and q are false otherwise always true.



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**Truth Table**

| p | q | $p \vee q$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

## 4. Implication or Conditional ($\rightarrow$)

The implication $p \rightarrow q$ (read as $p$ implies $q$) is false when $p$ is true and $q$ is false otherwise always true.

**Truth Table**

| p | q | $p \rightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

- $p$ is called antecedent or hypothesis.
- $q$ is called consequent or conclusion.
- $p \rightarrow q$ is also read as

If $p$, then q
$p$ implies $q$
$p$ only if $q$
$q$, if $p$
$p$ is sufficient for $q$
$q$ is necessary for $p$

### Conditionals Related with a Given Conditional

(i) The **converse** of $p \rightarrow q$ is $q \rightarrow p$

(ii) The **inverse** of $p \rightarrow q$ is $\neg p \rightarrow \neg q$

(iii) The **contra positive** of $p \rightarrow q$ is $\neg q \rightarrow \neg p$

## 5. Biconditional or Double Implication ($\leftrightarrow$)

The double implication $p \leftrightarrow q$ (read as $p$ iff $q$) is true when both $p$ and $q$ are true or both are false otherwise always false.

$p \leftrightarrow q$ read as
$p$ iff $q$
If $p$ then $q$ and if $q$ then $p$
$p$ if and only if $q$

**Truth Table**

| $p$ | $q$ | $p \leftrightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | T |



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## Logical Equivalences

Compound propositions that have same truth values are logically equivalent.

$\Leftrightarrow$ Stands for logically equivalent.

**Some Useful Equivalences:**

- $p \to q \sim (\neg p \vee q)$
- $p \to q \sim (\neg q \to \neg p)$

## Some Important Terms Related to Logic

| TERM | DEFINITION |
|---|---|
| Tautology | A statement which is always true is called tautology. |
| Absurdity or Contradiction | A statement which is always false, is called absurdity. |
| Contingency | A statement which can be true or false depending upon the truth values of the variables involved. |
| Quantifiers | The words or the symbols which give the idea of a quantity are called quantifiers. |
| Universal Quantifier | $\forall$ (For all) is called universal quantifier. |
| Existential Quantifier | $\exists$ (There exist) is called existential quantifier. |

## Link Between Set Theory and Logic

Let $p \equiv P$ and $q \equiv Q$

| LOGICAL STATEMENT | EQUIVALENCE IN SET THEORY |
|---|---|
| $\neg p$ | $P^c$ |
| $p \wedge q$ | $P \cap Q$ |
| $p \vee q$ | $P \cup Q$ |
| $p \to q$ | $P' \cup Q$ |
| $p \leftrightarrow q$ | $P = Q$ |
| $\neg(p \wedge q) = \neg p \vee \neg q$<br>$\neg(p \vee q) = \neg p \wedge \neg q$ | $(P \cap Q)' = P' \cup Q'$<br>$(P \cup Q)' = P' \cap Q'$ |

## Relations

| NAME | DEFINITION | EXAMPLE /REMARK |
|---|---|---|
| Cartesian Product | The Cartesian product of any two sets $A$ & $B$ is defined as $A \times B = \{(x,y) \mid x \in A \wedge y \in B\}$, Where $(x,y)$ is called an ordered pair. | Let $A = \{a,b\}$ & $B = \{1,2\}$ then $A \times B = \{(a,1),(a,2),(b,1),(b,2)\}$ |
| Binary Relation/Relation | Any subset of the Cartesian product $A \times B$ is called a relation from $A$ to $B$. | From above example $r_1 = \{(a,1),(b,2)\}$ |
| Domain of a Relation | The set of the first elements of the ordered pairs of a relation is called domain. | Dom of $r_1 = \{a,b\}$ |



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| | | |
|---|---|---|
| **Range of a Relation** | The set of the second elements of the ordered pairs of a relation is called range. | Range of $r_1 = \{1,2\}$ |
| **Inverse Relation** | If we interchange the domain and the range of every ordered pair of a relation then the new relation formed is the inverse of the given. | $r_1^{-1} = \{(1,a),(2,b)\}$ |

**Note:**

$A \times B \neq B \times A$ since $(x,y) \neq (y,x)$

## Functions

Let $A$ & $B$ are two non-empty sets then $f$ is a function from $A$ to $B$ if,

(i) $f \subseteq A \times B$

(ii) $Domf = A$

(iii) First elements of no two pairs of $f$ are equal.

e.g. $A = \{a,b\}$ and $B = \{1,2\}$, $r_1 = \{(a,1),(b,2)\}$ is a function from $A$ to $B$.

## Types of Functions

| NAME | DEFINITION | EXAMPLE/REMARK |
|---|---|---|
| **Into function** | A function $f: A \to B$ is called into if range $f \subset B$ | For $A = \{a,b\}$ & $B = \{1,2\}$, $r = \{(a,1),(b,1)\}$ is an into function from $A$ to $B$. |
| **Onto function/ Surjective function** | A function $f: A \to B$ is called onto if range $f = B$. | For $A = \{a,b\}$ & $B = \{1,2\}$, $r = \{(a,1),(b,2)\}$ is an onto function from $A$ to $B$. |
| **One-to-one function / Injective function** | A function $f: A \to B$ is called one-to-one function if second elements of no two of its ordered pairs are equal. | For $A = \{a,b\}$ & $B = \{1,2\}$, $r = \{(a,1),(b,2)\}$ is one-to-one function from $A$ to $B$. |
| **Bijective function** | A function which is both one-to-one and onto is called bijective function. | For $A = \{a,b\}$ & $B = \{1,2\}$, $r = \{(a,1),(b,2)\}$ is bijective function from $A$ to $B$. |

## Some important functions

| NAME | DEFINITION | EXAMPLE/REMARK |
|---|---|---|
| **Linear function** | A function of the form $\{(x,y) \mid y = mx + c\}$ is called linear. | $y = 2x + 1$ is a linear function. |
| **Quadratic function** | The function of the form $\{(x,y) \mid y = ax^2 + bx + c\}$ is called quadratic function. | $y = 3x^2 - 4x + 1$ is a quadratic function. |



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## Inverse function

The inverse of a function $f$ is denoted by $f^{-1}$ and can be obtained by interchanging the elements of the ordered pairs involved in the function.

**Do you know:**
- The graph of a linear function is always a straight line.
- The graph of a quadratic function is always a parabola.

## Remember that

Inverse of a function may or may not be a function.
Inverse of a line is line.

## Unary Operation

An operation which when performed on a single number yields another number of the same or a different system is called a unary operation.
For example square, square root, cube etc.

## Binary Operation

Binary operation denoted as ※ on a set $G$ is a function from the set $G \times G$ to $G$

## Properties of Binary* Operation

Let $S$ be a non-empty set and * is a defined as binary operation on $S$.

| NAME OF THE PROPERTY | DEFINITION |
|---|---|
| Closure Property | A non-empty set S is called closed w.r.t. * if $\forall a,b \in S \Rightarrow a*b \in S$ |
| Commutative Property | $\forall a,b \in S \Rightarrow a*b = b*a$ |
| Associative Property | $\forall a,b,c \in S \Rightarrow a*(b*c) = (a*b)*c$ |
| Identity Element | An element $e \in S$ is called identity w.r.t. * if $a*e = e*a = a$ |
| Inverse Property | For any element $a \in S$, $\exists a' \in S$ such that $a*a' = a'*a = e$, then $a$ and $a'$ are called inverse of each other w.r.t. *. |

## Residue Classes Modulo n

Let $n \in N$, then 0, 1, 2,…,$n-1$ are called residue and set $\{0,1,2,...,n-1\}$ is called residue classes under modulo $n$.
e.g. $\{\bar{0},\bar{1},\bar{2},\bar{3}\}$ are the residue classes under modulo 4.

- The following table* represents the multiplication of residue classes under modulo 5.

| ⊗ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 |
| 2 | 2 | 4 | 1 | 3 |
| 3 | 3 | 1 | 4 | 2 |
| 4 | 4 | 3 | 2 | 1 |

- The following table represents the addition of the residue classes modulo 7.

* In chapter 1 we have discussed about binary and unary operations.
* Table is called Cayley's table named for the British mathematician Arthur Cayley (1821-1895)



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| $\oplus$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 0 |
| 2 | 2 | 3 | 4 | 5 | 6 | 0 | 1 |
| 3 | 3 | 4 | 5 | 6 | 0 | 1 | 2 |
| 4 | 4 | 5 | 6 | 0 | 1 | 2 | 3 |
| 5 | 5 | 6 | 0 | 1 | 2 | 3 | 4 |
| 6 | 6 | 0 | 1 | 2 | 3 | 4 | 5 |

- The following table represent the multiplication of cube roots of unity.

| $\otimes$ | 1 | $\omega$ | $\omega^2$ |
|---|---|---|---|
| 1 | 1 | $\omega$ | $\omega^2$ |
| $\omega$ | $\omega$ | $\omega^2$ | 1 |
| $\omega^2$ | $\omega^2$ | 1 | $\omega$ |

- The following table describe that which properties are satisfied by the specified set of numbers.

| PROPERTY/ SET OF NUMBERS | | NATURAL ($\mathbb{N}$) | WHOLE ($W$) | INTEGERS ($\mathbb{Z}$) | RATIONAL ($\mathbb{Q}$) | REAL ($\mathbb{R}$) |
|---|---|---|---|---|---|---|
| Closure | $\oplus$ | ✓ | ✓ | ✓ | ✓ | ✓ |
| | $\otimes$ | ✓ | ✓ | ✓ | ✓ | ✓ |
| Associative | $\oplus$ | ✓ | ✓ | ✓ | ✓ | ✓ |
| | $\otimes$ | ✓ | ✓ | ✓ | ✓ | ✓ |
| Identity | $\oplus$ | × | ✓ | ✓ | ✓ | ✓ |
| | $\otimes$ | ✓ | ✓ | ✓ | ✓ | ✓ |
| Inverse | $\oplus$ | × | × | ✓ | ✓ | ✓ |
| | $\otimes$ | × | × | × | × | × |
| Commutative | $\oplus$ | ✓ | ✓ | ✓ | ✓ | ✓ |
| | $\otimes$ | ✓ | ✓ | ✓ | ✓ | ✓ |



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## Groups

Let $S$ be a non-empty set and * is a defined as binary operation on $S$.

**Groupoid:** If $S$ is a non-empty set closed w.r.t * or a binary operation * is defined on $S$ then it is called Groupoid,

$(\mathbb{N},+)$, $(\mathbb{R},+)$, $(\mathbb{Z},+)$, $(\mathbb{Q},+)$ are some Groupoids.

↓

**Semi group:** If a Groupoid holds associative law w.r.t * then it becomes semi group, $(\mathbb{N},+)$, $(\mathbb{R},+)$, $(\mathbb{Z},+)$, $(\mathbb{Q},+)$ are some semi groups.

↓

**Monoid:** If a semi group has identity elements w.r.t * then it becomes Monoid. $(\mathbb{R},+)$, $(\mathbb{Z},+)$, $(\mathbb{Q},+)$ are some Monoids.

↓

**Group:** If in a Monoid there exist inverse of each element w.r.t * then it is called group $(\mathbb{R},+)$, $(\mathbb{Z},+)$, $(\mathbb{Q},+)$ are some Groups.

↙ ↘

**Abelian Group:** If a group holds commutative law w.r.t * then it becomes abelian group.

$(\mathbb{Q},+), (\mathbb{R},+), (\mathbb{Z},+)$

are some Abelian Groups.

**Non-Abelian Group:** If a group does not holds commutative law w.r.t * then it called non-abelian group.

Set $\left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix}, ad-bc \neq 0\right\}$ is some non-abelian Group.

**Do you know?**
A group w.r.t addition cannot be group w.r.t multiplication



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| SUMMARY | |
|---|---|
| Groupoid | $(N,+)$, $(N,\times)$, $(W,+)$, $(W,\times)$, (Z,+), (Z,×), (E,+), (E,×), $(O,\times)$, (Q,+), (Q,×), (R,+), (R,×), (C,+), (C,×) |
| Semi Group | $(N,+)$, $(N,\times)$, $(W,+)$, $(W,\times)$, (Z,+), (Z,×), (E,+), (E,×), $(O,\times)$, (Q,+), (Q,×), (R,+), (R,×), (C,+), (C,×) |
| Monoid | $(N,\times)$, $(W,+)$, $(W,\times)$, (Z,+), (Z,×), (E,+), $(O,\times)$, (Q,+), (Q,×), $(R,+)$, (R,×), (C,+), (C,×) |
| Group | (Z,+), $(E,+)$, $(Q,+)$, $(R,+)$, $(C,+)$ |
| Abelian Group | (Z,+), $(E,+)$, $(Q,+)$, $(R,+)$, $(C,+)$ |
| Non-Abelian Group | $\left(\left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in R \wedge ad-bc \neq 0\right\},\times\right)$ |

## Some Concept Related To Groups

| | |
|---|---|
| Order of a Group | The number of elements of a group is called order of the group. |
| Finite & Infinite Groups | A group is called finite if its order is finite otherwise it is called infinite group. |
| Uniqueness of Identity | In a group the identity element is unique. |
| Uniqueness of Inverse | In a group inverse of each element w.r.t. * is unique. |
| Reversal Law of Inverses | $\forall a,b \in S$ where S is a group then $(ab)^{-1} = b^{-1}a^{-1}$ |
| Idempotent Element | Let S be a group & $a \in S$ then $a$ is called idempotent if $a * a = a$. |

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# 3 CHAPTER MATRICES AND DETERMINANTS

## Introduction

| NAME OF MATHEMATICIAN | WORK IN MATRICES |
|---|---|
| James Sylvester (1814-1897) | Used the word matrix first time |
| Arthur Cayley (1821-1895) | Used in linear transformations |
| Seki Kowa (1642-1708) | Concept of determinants |
| Gottfried Wilhelm Leibniz (1646-1716) | Invention of determinants |
| Gabriel Cramer (1704-1752) | Used determinants for solving system of linear equations |

## Matrix

A rectangular array of numbers enclosed by a pair of brackets is called a matrix.

OR

Arrangement of numbers in rows and columns is called a matrix.

## Rows

The horizontal lines of numbers are called rows of a matrix.

## Columns

The vertical lines of numbers are called columns of a matrix.

## Entries or Elements

The numbers used in rows or columns are said to be the entries or elements of the matrix.

## Order of a Matrix

Order of a matrix is defined as:

**Number of rows** × **Number of columns** (read as number of rows by number of columns)

## General Definition of a Matrix

Generally a bracketed rectangular array of $m \times n$ elements $a_{ij}\ (i = 1,2,3,\ldots,\ m, j = 1,2,3,4,\ldots,n)$, arranged in $m$ rows and $n$ columns is called a $m$ by $n$ matrix (written as $m \times n$ matrix).

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

Where $a_{11}, a_{22}, a_{33}, \ldots, a_{mn}$ are called main or leading or principal diagonal elements.



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## Types of Matrices

| NAME OF MATRIX | DEFINITION | EXAMPLE |
|---|---|---|
| **Row matrix (row vector)** | A matrix, which has only one row (*i.e.* $1 \times n$ matrix) is said to be a row matrix or row vector. | $\begin{bmatrix}1 & 2 & 0 & 5\end{bmatrix}_{1\times4}$ |
| **Column matrix (column vector)** | A matrix, which has only one column (*i.e.* an $m \times 1$ matrix) is said to be a column matrix or column vector. | $\begin{bmatrix}2\\3\\4\\5\end{bmatrix}_{4\times1}$ |
| **Square matrix** | The matrix, which has the same number of rows and columns, is called a square matrix.<br>OR<br>If $m = n$ then the matrix of order $m \times n$ is said to be a square matrix of order $n$ or $m$. | $[7]_{1\times1}$<br>$\begin{pmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{pmatrix}_{2\times2}$ |
| **Rectangular matrix** | A matrix, whose number of rows is not equal to number of columns, is called a rectangular matrix.<br>*OR*<br>If $m \neq n$, then the matrix is called rectangular matrix of order $m \times n$. | $\begin{bmatrix}2 & -1 & 7\\9 & 5 & -2\end{bmatrix}_{2\times3}$ |
| **Diagonal matrix** | Let $A = [a_{ij}]_{n\times n}$ be a square matrix of order n. If $a_{ij} = 0$ for all $i \neq j$ and at least one $a_{ij} \neq 0$ for $i = j$ | $\begin{bmatrix}1 & 0 & 0\\0 & 9 & 0\\0 & 0 & 7\end{bmatrix}$ |
| **Scalar matrix** | Let $A = [a_{ij}]_{n\times n}$ be a square matrix of order n. If $a_{ij} = 0$ for all $i \neq j$ and $a_{ij} = k$ for all $i = j$, where k is any non zero scalar. | $\begin{pmatrix}2 & 0 & 0\\0 & 2 & 0\\0 & 0 & 2\end{pmatrix}$ |
| **Unit matrix *or* Identity matrix** | Let $A = [a_{ij}]_{n\times n}$ be a square matrix of order n. If $a_{ij} = 0$ for all $i \neq j$ and $a_{ij} = 1$ for all $i = j$. | $I_2 = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ |
| **Null matrix *or* Zero matrix** | A square or rectangular matrix whose each element is zero is called a null or zero matrix. | $\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$ |

- Unit matrix is both diagonal & scalar matrix.
- Unit matrix is multiplicative identity of matrices.

Null matrix is additive identity of matrices.



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| | | |
|---|---|---|
| **Equal Matrices** | Two matrices of the same order are said to be equal if their corresponding entries are equal. | $\begin{pmatrix}1 & 7 & 2\\ 3 & 4 & 8\end{pmatrix}=$ $\begin{pmatrix}2-1 & 4+3 & 2\\ 2+1 & 2+2 & 8\end{pmatrix}$ |
| **Sub Matrix** | Matrix obtained by deleting some rows or columns from an existing matrix are said to be sub-matrix of the original matrix. | |
| **Idempotent Matrix** | A Square matrix $A$ is said to be idempotent if $A^2 = A$ | $A=\begin{bmatrix}2 & -3 & -5\\ -1 & 4 & 5\\ 1 & -3 & -4\end{bmatrix}$ |
| **Nilpotent Matrix** | A Square matrix $A$ is said to be nilpotent if $A^P = O$ | $A=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$ |
| **Involutory Matrix** | A Square matrix $A$ is said to be involutory if $A^2 = I$ | $A=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ |
| **Periodic Matrix** | A square matrix $A$ is set to be periodic matrix of period $P$ if $P$ least positive integer such that $A^{P+1} = A$ | $A=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$ Is periodic with period 2. |
| **Orthogonal Matrix** | A square matrix $A$ of order $n$ is said to be orthogonal if $AA^t = A^tA = I_n$ | $A=\begin{bmatrix}\cos A & \sin A\\ -\sin A & \cos A\end{bmatrix}$ |

## Matrix Operations & Their Properties

| OPERATION | DEFINITION | PROPERTIES |
|---|---|---|
| **Addition of Matrices** | Two matrices are conformable for addition if they are of the same order. | **(i)** $A+B=B+A$ (commutative property)<br>**(ii)** $A+(B+C)=(A+B)+C$ (associative property)<br>**(iii)** $A+O=O+A=A$ (O is additive identity) |
| **Negative of a Matrix** | If A is a given matrix, then the matrix obtained by writing the negative of each entry in place of the each entry of A is called the negative of A. | |
| **Scalar Multiplication** | If $A=[a_{ij}]$ is an $m\times n$ matrix and $k$ is a scalar, then the product of $k$ (any non zero constant) and $A$ denoted by $kA$ is the matrix formed by multiplying each entry of $A$ by $k$, that is: $kA = [ka_{ij}]$ | **(i)** $c(A+B)=cA+cB$<br>**(ii)** $(c+d)A=cA+dA$<br>**(iii)** $c(AB)=(cA)B=A(cB)$<br>(Distributive properties w.r.t. scalar multiplication) |



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| | | |
|---|---|---|
| **Subtraction of Matrices** | If $A=\left[a_{ij}\right]$ and $B=\left[b_{ij}\right]$ are matrices of order $m\times n$ then we define subtraction of $B$ from $A$ as: $A-B=A+(-B)$ | |
| **Multiplication of two matrices** | Two matrices $A$ and $B$ are said to be conformable for the product $AB$ if the number of columns of A is equal to the number of rows of $B$. | **(i)** $A(BC)=(AB)C$ (Associative property)<br>**(ii)** $AB\neq BA$ (not commutative in general)<br>**(iii)** $IA=AI=A$ ($I$ is multiplicative identity)<br>**(iv)** A(B+C)=AB+AC (distributive property of multiplication over +)<br>**(v)** $A_{m\times n}B_{n\times p}=(AB)_{m\times p}$ |
| **Trace of a matrix** | For a square matrix of order $n\times n$ ,the trace of the matrix $a_{11}+a_{22}+a_{33}+\cdots+a_{nn}$ | The trace of $\begin{pmatrix}1&9&2\\3&8&5\\4&6&0\end{pmatrix}$ is 9. |

## Transpose of a Matrix

If $A$ is a matrix of order $m\times n$ then an $n\times m$ matrix obtained by interchanging the rows and columns of $A$, is called the transpose of $A$. denoted by $A'$ or $A'$.

For example

$$A=\begin{bmatrix}2&7\\1&5\\5&4\end{bmatrix}\Rightarrow A'=\begin{bmatrix}2&1&5\\7&5&4\end{bmatrix}$$

## Properties of Transpose of Matrices

(i) $\left(A'\right)'=A$

(ii) $\left(A+B\right)'=A'+B'$

(iii) $\left(A+B+C\right)'=A'+B'+C'$

(iv) $\left(cA\right)'=c\left(A'\right)$ where $c$ is a non zero scalar.

(v) $\left(AB\right)'=B'A'$

(vi) $\left(ABC\right)'=C'B'A'$

## 1. The Determinant of a 1×1 Matrix

$A=\left[a_{11}\right]$ is defined as $\left|A\right|=\left|a_{11}\right|=a_{11}$

## 2. Determinant of a 2×2 Matrix

The determinant of a matrix is denoted by enclosing its square array between vertical bars instead of brackets.

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- If $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $|A| = ad - bc$
- If $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $adjA=\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

## 3. Minor of an Element of a Square Matrix

Given an $n\times n$ matrix $A=(a_{ij})_{n\times n}$ if we delete $i$th row and $j$th column of A, we get an $(n-1)\times(n-1)$ matrix. The determinant of this matrix is called the minor of the element $a_{ij}$ and is denoted by $M_{ij}$.

## 4. Cofactor of an Element of a Square Matrix

The cofactor of an element $a_{ij}$ of a square matrix A is denoted by $A_{ij}$ and is defined as

$A_{ij}=(-1)^{i+j}M_{ij}$

## 5. Adjoint of a Matrix of Order $n\geq 3$

If A is a square matrix, then the matrix obtained by taking the cofactor of each element of A, replacing them correspondingly and then taking transpose of that, is called adjoint of A.

## 6. The Determinant of a Square Matrix of Order $n\geq 3$

The determinant of a square matrix of order $n\geq 3$ is the sum of the product of each element of row (or column) and its cofactor. *i.e.*

$|A| = a_{11}A_{11} + a_{12}A_{12} + ... + a_{1n}A_{1n}$ or

$|A| = a_{21}A_{21} + a_{22}A_{22} + ... + a_{2n}A_{2n}$ or

$|A| = a_{31}A_{31} + a_{32}A_{32} + ... + a_{3n}A_{3n}$

## Properties of Determinants

**(i)** For a square matrix $A$, $|A|=|A'|$.

**(ii)** If two rows (or columns) of a determinant are interchanged, the value of determinant is multiply by –1.

**(iii)** If all the entries in any row (or column) are zero, the value of the determinant is zero.

**(iv)** If any two rows (or columns) of a determinant are identical, the value of the determinant is zero.

**(v)** If any row (or column) of a determinant is multiplied by a non-zero number k, the value of the new determinant becomes equal to k times the value of original determinant. $|kA|=k^n|A|$, where $n$ is the order of $A$.

**(vi)** If any row (or column) of a determinant consists of two terms, it can be written as the sum of two determinants.

**(vii)** If any row (or column) of a determinant is multiplied by a non-zero number k and the result is added to the corresponding entries of another row (or column), the value of the determinant does not change.

**(viii)** The determinant of a triangular matrix (upper, lower, or diagonal) is equal to the product of the elements of the main diagonal.

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## Singular Matrix

A square matrix A is said to be singular if $|A| = 0$

## Non Singular Matrix

A square matrix $A$ is said to be non-singular if $|A| \neq 0$

## Inverse of a Matrix

$$A^{-1} = \frac{1}{|A|} adjA \text{ if } |A| \neq 0$$

## Properties of Inverse of Matrix

(i) $\left(A^{-1}\right)^{-1} = A$

(ii) $\left(A'\right)^{-1} = \left(A^{-1}\right)'$

(iii) $(AB)^{-1} = B^{-1}A^{-1}$

(iv) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

(v) $\left(A^n\right)^{-1} = \left(A^{-1}\right)^n$, for any $n \in \mathbb{Z}^+$

(vi) $(kA)^{-1} = k^{-1}A^{-1}$, $k$ is any non Zero scalar.

## Classification of Matrices w.r.t. Operations

| MATRIX OPERATION/NAME | DEFINITION | EXAMPLE |
|---|---|---|
| Symmetric Matrix | For a square matrix A if $A' = A$, then A is called symmetric matrix | $\begin{pmatrix} 1 & 3 & 5 \\ 3 & 9 & 6 \\ 5 & 6 & 8 \end{pmatrix}$ |
| Skew Symmetric or Anti Symmetric | For a square matrix A if $A' = -A$, then A is called skew symmetric matrix | $\begin{pmatrix} 0 & 1 & -3 \\ -1 & 0 & -7 \\ 3 & 7 & 0 \end{pmatrix}$ |
| Hermitian Matrix | A square matrix A with complex entries is called hermitian if $\left(\overline{A}\right)' = A$. . | $\begin{pmatrix} 1 & 1+i \\ 1-i & 4 \end{pmatrix}$ |
| Skew Hermitian Matrix | A square matrix A with complex entries is called skew hermitian if $\left(\overline{A}\right)' = -A$ | $\begin{pmatrix} 0 & 2+5i \\ -2+5i & 0 \end{pmatrix}$ |
| Upper Triangular Matrix | A square matrix A=$\left[a_{ij}\right]_{n\times n}$ is called an upper triangular if $a_{ij} = 0 \ \forall \ i > j$ | U=$\begin{pmatrix} 1 & 7 & 2 \\ 0 & -9 & 7 \\ 0 & 0 & 0 \end{pmatrix}$ |



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| | | |
|---|---|---|
| **Lower triangular matrix** | A square matrix $A=\left[a_{ij}\right]_{n\times n}$ is said to be a lower triangular matrix if $a_{ij}=0\ \forall\ i<j$ | $L=\begin{pmatrix}2&0&0\\2&-3&0\\-1&6&9\end{pmatrix}$ |
| **Triangular matrix** | A square matrix A is named as triangular whether it is upper triangular or lower triangular. | In above examples U & L are triangular matrices. |

Diagonal matrices are both upper triangular and lower triangular.

## Elementary Row and Column Operations on a Matrix

Usually a given system of linear equations is reduced to a simple equivalent system by applying in turn a finite number of elementary operations, which are stated as below:

1. Interchanging two equations.
2. Multiplying an equation by a non-zero number.
3. Adding a multiple of one equation to another equation.

The system of linear equation involving the same variable, are equivalent if they have the same solution.

### (a) Elementary Row Operation on a Matrix

The following elementary row operations are applied to matrices to obtain equivalent matrix.

(i) Interchanging two rows. $R_i \leftrightarrow R_j$

(ii) Multiplying a row by a non-zero number.

$kR_i \leftrightarrow R_i'$

(iii) Adding a multiple of one row to another row.

$R_i + kR_j = R_i'$

Matrices $A$ and $B$ are equivalent it can be obtained by applying in turn a finite number of operations on $A$.

### (b) Elementary Column Operation on a Matrix

Following are the elementary column operations that are used to obtain equivalent matrices.

(i) Interchanging two columns. $C_i \leftrightarrow C_i'$

(ii) Multiplying a column by a non-zero number. $kC_i \leftrightarrow C_i'$

(iii) Adding a multiple of one column to another column. $C_i + kC_j = C_i'$

## Echelon and Reduced Echelon Forms of Matrices

In any non-zero row of a matrix the first non-zero entry is called the leading entry of that row. The zeros before the leading entry of a row are named as the leading zero entries of the row.

### (a) Echelon Form of a Matrix

An $m\times n$ matrix $A$ is called in (row) echelon form if:

(i) In each successive non-zero row, the number of zeros before the leading entry is greater than the number of such zeros in the preceding row.

(ii) Every non-zero row in A precedes every zero row (if any).

(iii) The first non zero entry (or leading entry) in each row is 1.



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For example the matrices $A=\begin{bmatrix}1 & 0 & 6\\ 0 & 1 & -4\\ 0 & 0 & 1\end{bmatrix}$ and $B=\begin{bmatrix}1 & 1 & 2\\ 0 & 1 & -1\\ 0 & 0 & 0\end{bmatrix}$ are in Echelon form.

## (b) Reduced Echelon Form of a Matrix

An $m\times n$ matrix $A$ is said to be in reduced (row) echelon form if it is in (row) echelon form and if the first non-zero entry (or leading entry) in $R_i$ lies in $C_j$ then all other entries of $C_j$ are zero.

For example the matrices $A=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ and $B=\begin{bmatrix}1 & 0 & 3\\ 0 & 1 & -1\\ 0 & 0 & 0\end{bmatrix}$ are in reduced Echelon form.

## Rank of a Matrix

Let A be a non-zero matrix. If 'r' is the number of non-zero rows when it is reduced to the reduced echelon form, then 'r' is called the (row) rank of the matrix A.

## System of Linear Equations

| NON-HOMOGENEOUS SYSTEM | HOMOGENEOUS SYSTEM |
|---|---|
| • An equation of the form $ax+by+cz=d$ $a\neq 0,b\neq 0,c\neq 0,d\neq 0$ is called non-homogeneous equation in three variables.<br>• $\left.\begin{aligned}a_1x+b_1y+c_1z&=d_1\\ a_2x+b_2y+c_2z&=d_2\\ a_3x+b_3y+c_3z&=d_3\end{aligned}\right\}$ is called a system of non-homogeneous equations in three variables where , $d_1$, $d_2$ and $d_3$ cannot zero simultaneously.<br>• The matrix form of the system is $\begin{pmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}d_1\\ d_2\\ d_3\end{pmatrix}$ can be written as $AX=D$ where $A=\begin{pmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{pmatrix}$, $X=\begin{pmatrix}x\\ y\\ z\end{pmatrix}$ , $D=\begin{pmatrix}d_1\\ d_2\\ d_3\end{pmatrix}$. | • An equation of the form $ax+by+cz=0$ is called homogeneous equation in three variables.<br>• $\left.\begin{aligned}a_1x+b_1y+c_1z&=0\\ a_2x+b_2y+c_2z&=0\\ a_3x+b_3y+c_3z&=0\end{aligned}\right\}$ is called system of homogeneous equations in three variables.<br>• The matrix form of the system is $\begin{pmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$ can be written as $AX=O$ where $A=\begin{pmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{pmatrix}$, $X=\begin{pmatrix}x\\ y\\ z\end{pmatrix}, O=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$. |



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## Crammer's Rule

Consider the system of equations,

$$\left.\begin{aligned} a_1x+b_1y+c_1z&=d_1\\ a_2x+b_2y+c_2z&=d_2\\ a_3x+b_3y+c_3z&=d_3 \end{aligned}\right\}$$ By using crammer's rule the solution of the system is defined as,

$$x=\frac{\begin{vmatrix} d_1 & b_1 & c_1\\ d_2 & b_2 & c_2\\ d_3 & b_3 & c_3 \end{vmatrix}}{|A|},\ y=\frac{\begin{vmatrix} a_1 & d_1 & c_1\\ a_2 & d_2 & c_2\\ a_3 & d_3 & c_3 \end{vmatrix}}{|A|},\ z=\frac{\begin{vmatrix} a_1 & b_1 & d_1\\ a_2 & b_2 & d_2\\ a_3 & b_3 & d_3 \end{vmatrix}}{|A|},\ |A|\neq 0$$

## Some Important Properties

**(i)** The sum and difference of two symmetric matrices is always a symmetric matrix.

**(ii)** The sum and difference of two skew–symmetric matrices is skew symmetric matrix.

**(iii)** If a matrix 'A' is symmetric or skew symmetric then $kA$ is also symmetric or skew symmetric matrix respectively. Where $k$ is any non-zero constant.

**(iv)** If $A$ is a square matrix then $A+A'$ is symmetric and $A-A'$ is skew symmetric matrix.

**(v)** If two matrices $A$ and $B$ are symmetric or skew symmetric and $AB=BA$ then both $AB$ and $BA$ are symmetric matrices.

**(vi)** Every square matrix is uniquely expressible as the sum of symmetric matrix and skew symmetric matrix.

**(vii)** If $A$ be any square matrix, then $AA'$ and $A'A$ are symmetric matrices.

**(viii)** If $A$ is symmetric matrix then $A^n, \forall n\in\mathbb{N}$ is also symmetric matrix.

**(ix)** If $A$ is skew symmetric matrix then $A^n, \forall n\in O^+$ is skew symmetric and $A^n, \forall n\in E^+$ is symmetric.

**(x)** If $A$ and $B$ are symmetric matrices of same order then

**(a)** $(AB+BA)$ is symmetric matrix.

**(b)** $(AB-BA)$ is skew symmetric matrix.

**(xi)** If A is non–singular matrix such that A is symmetric then $A^{-1}$ is also symmetric.

**(xii)** The sum and difference of two Hermitian matrices is always a Hermitian matrix.

**(xiii)** The sum and difference of two Skew Hermitian matrices is always Skew Hermitian.

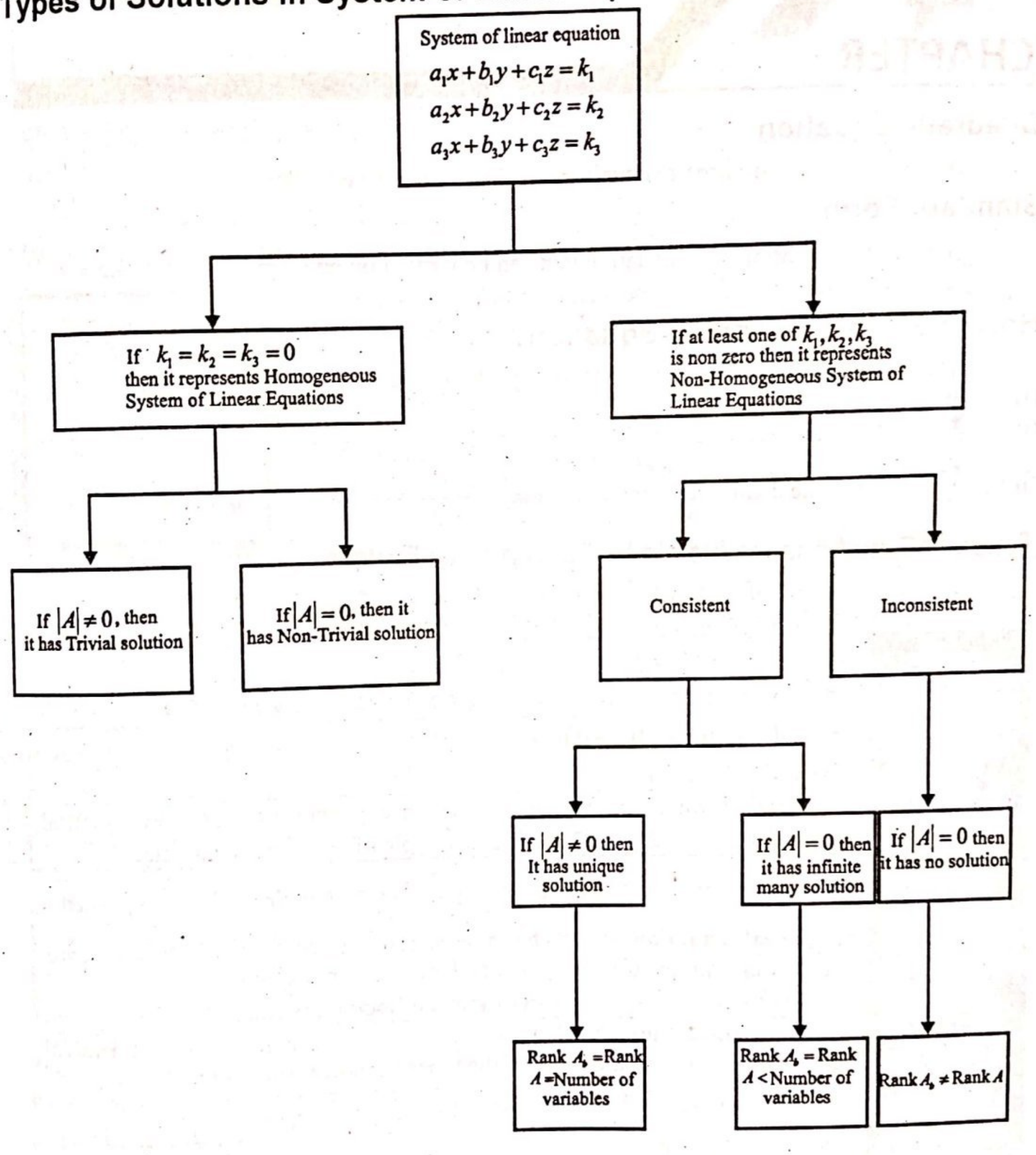
**(xiv)** If $A$ is a square matrix then $A+(\overline{A})'$, $A(\overline{A})'$ and $(\overline{A})'A$ are always Hermitian matrices.

**(xv)** If $A$ is a square matrix then $A-(\overline{A})'$ is always skew Hermitian matrix.



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## Types of Solutions in System of Linear Equations






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# 4 CHAPTER QUADRATIC EQUATION

## Quadratic Equation

A second degree polynomial equation is called quadratic equation.

## Standard Form

A quadratic equation is an equation that can be written in the form of $ax^2+bx+c=0$, where $a$, $b$ and $c$ are real numbers and $a \neq 0$.

## How to Solve a Quadratic Equation?

There are three basic techniques to solve a quadratic equation:

(i) By factorization

(ii) By completing squares, extracting square roots.

(iii) By applying the quadratic formula i.e. $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

**Do you know?**
The graph of a quadratic function represents a parabola opening either upwards or downwards

## Types of Equations Reducible to the Quadratic Equation

There are certain types of equations, which do not look to be of degree 2, but they can be reduced to the quadratic equation. Some of those forms are given below:

| | |
|---|---|
| **Type I** | $ax^{2n}+bx^n+c=0, a\neq 0$<br>Put $x^n=y$ and get the given equation reduced to quadratic equation in y. |
| **Type II** | $(x+a)(x+b)(x+c)(x+d)=k$<br>Where $a+b=c+d$ |
| **Type III** | Equations in which the variable occurs in exponent are called **exponential equations.** $2^{2x}-3.2^{x+2}+32=0$ is an example of exponential equation. |
| **Type IV** | An equation, which remains unchanged when $x$ is replaced by $\dfrac{1}{x}$ is called a **reciprocal equation.** In such an equation the coefficients of the terms equidistant from the beginning and end are equal in magnitude.<br>$ax^4+bx^3+cx^2+bx+a=0$ is an example of reciprocal equation. |
| **Type V** | Equations involving radical expression of the variable are called **radical equations.** Following are some of the types of radical equations.<br>$(ax^2+bx)+m\sqrt{ax^2+bx+c}=n$, $\sqrt{ax+b}+\sqrt{cx+d}=\sqrt{ex+f}$ |
| **Type VI** | $\dfrac{a}{x+b}+\dfrac{c}{x+d}=\dfrac{e}{x+f}$<br>$x\neq -b,-d,-f$ |
| **Type VII** | $ay+\dfrac{b}{y}=c, y\neq 0$ |



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## Three Cube Roots of Unity

The three cube roots of unity are; $1, \omega = \frac{-1+\sqrt{3}i}{2}, \omega^2 = \frac{-1-\sqrt{3}i}{2}$.

From the above results we can write

| | |
|---|---|
| $-1+\sqrt{3}i = 2\omega$ | $-1-\sqrt{3}i = 2\omega^2$ |
| $1-\sqrt{3}i = -2\omega$ | $1+\sqrt{3}i = -2\omega^2$ |
| $\frac{1-\sqrt{3}i}{2} = -\omega$ | $\frac{1+\sqrt{3}i}{2} = -\omega^2$ |

**Do you know?**
- Sum of cube roots of any real number is zero.
- Product of cube roots of any real number is real number itself.

## Properties of Cube Roots of Unity

- The sum of all the three cube roots of unity is zero *i.e.* $1+\omega+\omega^2=0 \Rightarrow 1+\omega=-\omega^2,\ 1+\omega^2=-\omega,\ \omega+\omega^2=-1$
- The product of all the three cube roots of unity is unity *i.e.* $\omega^3=1$
- Each complex cube root of unity is reciprocal / multiplicative inverse of other $\Rightarrow \frac{1}{\omega}=\omega^2, \quad \frac{1}{\omega^2}=\omega$
- Each complex cube root of unity is square of the other. i.e. $(\omega)^2=\omega^2, (\omega^2)^2=\omega$
- For any $n \in \mathbb{Z}$, $\omega^n$ is equivalent to one of the cube roots of unity.
- $\omega^n+\omega^{n+1}+\omega^{n+2}=0, \forall n \in Z$

## Four Fourth Roots of Unity

Four fourth roots of unity are $1,-1,i,-i$.

**Do you know?**
- Sum of four fourth roots of any positive integer is zero.
- Product of four fourth roots of any positive integer is the additive inverse of that integer.

## Properties of Four Fourth Roots of Unity

- Sum of all the four fourth roots of unity is zero. i.e. $1+(-1)+i+(-i)=0$
- Product of all the four fourth roots of unity is $-1$. i.e. $1.(-1).i.(-i)=-1$
- The real fourth roots of unity are additive inverse of each other.
- The complex/imaginary fourth roots of unity are conjugate of each other.

## Polynomial Function

A function of the form $P(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$, $a_n \neq 0$, where $n$ is a non-negative integer and the coefficients $a_n, a_{n-1}, \ldots, a_1, a_0$ are real numbers. It can be considered as a polynomial function of $x$.

## Degree of the Polynomial

The highest power of $x$ in a polynomial $P(x)$ is called the degree of polynomial

So degree of polynomial $P(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$, $a_n \neq 0$, is $n$.



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## Types of Polynomial on the Basis of Degree

| NAME | DEFINITION | EXAMPLE |
|---|---|---|
| Constant Polynomial | A polynomial whose degree is 0 | 4 |
| Linear Polynomial | A polynomial whose degree is 1 | $2x-1$ |
| Quadratic Polynomial | A polynomial whose degree is 2 | $3x^2-2x+1$ |
| Cubic Polynomial | A polynomial whose degree is 3 | $4x^3+3$ |
| Quartic Polynomial | A polynomial whose degree is 4 | $2x^4-5x^2+9$ |

## Types of Polynomial on the Basis of Number of Terms

| NAME | DEFINITION | EXAMPLE |
|---|---|---|
| Monomial | A polynomial having only one term | $2x$ |
| Binomial | A polynomial having two terms | $3x-y$ |
| Trinomial | A polynomial having three terms | $2x^2+3x-1$ |

**Remember that:**
0 is a polynomial with arbitrary degree

## Theorems on Polynomials

### Remainder Theorem

When a polynomial $P(x)$ of degree $n(n\geq 1)$ is divided by $x-a$,
Then the remainder is a constant $P(a)$.

**Note:**

Dividend = (Divisor) (Quotient) + Remainder.

### Factor Theorem

The polynomial $x-a$ is a factor of the polynomial $P(x)$ of degree $n(n\geq 1)$
If and only if $P(a)=0$.

## Synthetic Division

### Outline of the Method

(i) Write down the coefficients of the dividend $f(x)$ from left to right in decreasing order of powers of $x$. Insert 0 for any missing term(s).

(ii) To the left of the first line, write the value of a from the divisor $(x-a)$.

(iii) Use the following patterns to write the 2nd & 3rd line:
Vertical pattern $(\downarrow)$ add terms
Diagonal pattern $(\nearrow)$ multiply by $a$.

## Relations Between the Roots and the Coefficients of a Quadratic Equation

If $\alpha,\beta$ are roots of quadratic equation $ax^2+bx+c=0$ then

$$\text{Sum of the roots} = S = -\frac{b}{a} = -\frac{\text{coefficient of } x}{\text{coefficient of } x^2}$$

$$\text{Product of the roots} = P = \frac{c}{a} = \frac{\text{constant term}}{\text{coefficient of } x^2}$$

$$\text{Difference of roots} = \sqrt{S^2-4P}$$

For an equation $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$, $a_n \neq 0$,



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Sum of roots $= -\frac{a_{n-1}}{a_n}$

Product of roots $= (-1)^n \frac{a_0}{a_n}$

## Formation of an Equation Whose Roots Are Given

Let $S$ = sum of the roots and $P$ = product of the roots of a quadratic equation

Then the required equation will be; $x^2 - Sx + P = 0$

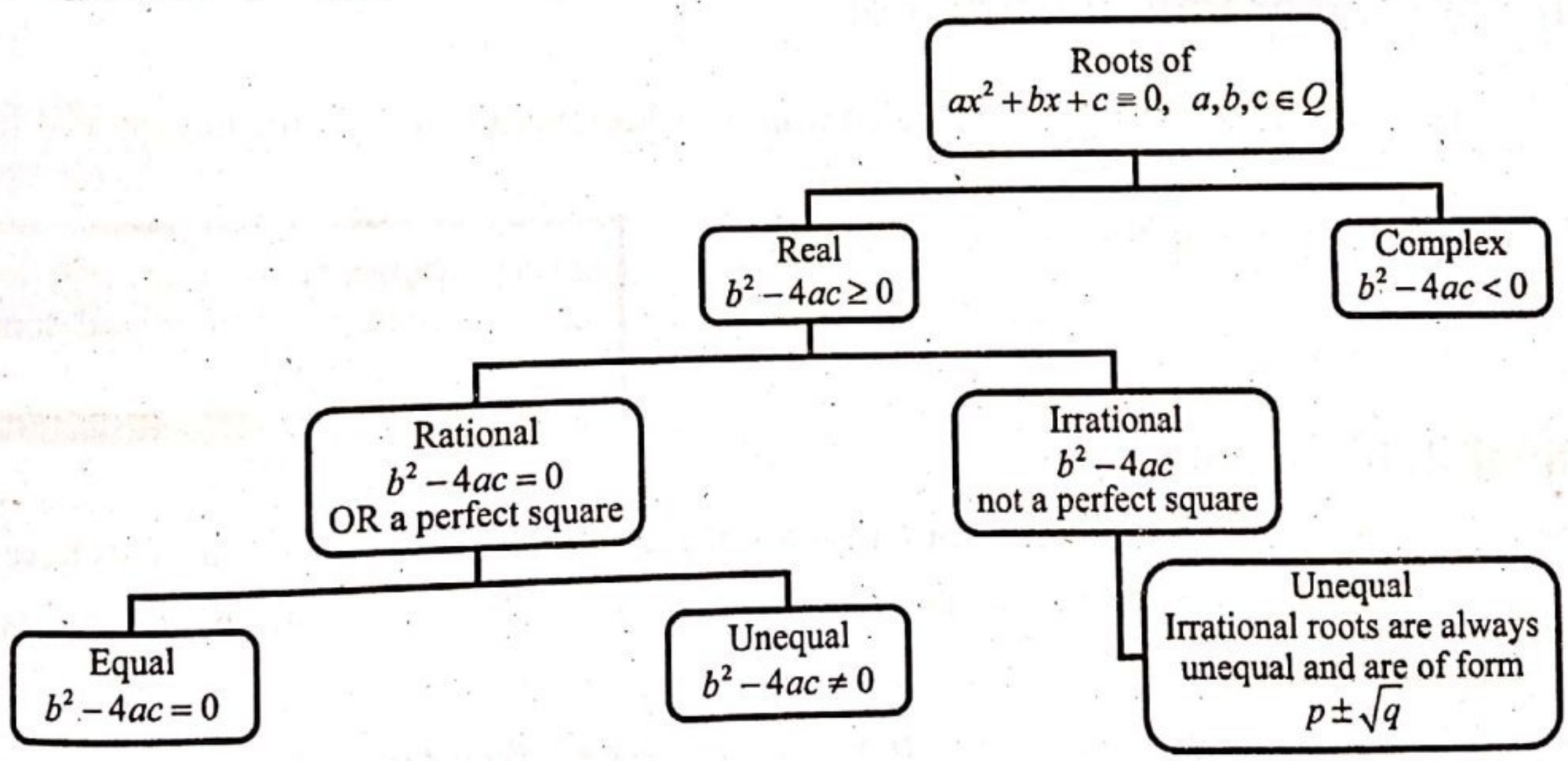
**Some Useful Results:**

- $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$
- $a^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$
- $\alpha^3 - \beta^3 = (\alpha + \beta)\left[(\alpha + \beta)^2 - \alpha\beta\right]$
- $\alpha^2 - \beta^2 = (\alpha + \beta)\sqrt{(\alpha + \beta)^2 - 4\alpha\beta}$

## Nature of Roots of a Quadratic Equation

The roots of quadratic equation $ax^2 + bx + c = 0$ depends on the value of the expression $b^2 - 4ac$, which is called its **Discriminant**

i.e. disc$= b^2 - 4ac$




## System of Two Equations Involving Two Variables

To determine the value of two variables, we need a pair of equations. Such a pair of equation is called a system of simultaneous equations.

No specific rule can be laid down for the solution of such equation. Simply, one of the variables is eliminated and the resulting equation in one variable is solved.

## Homogeneous Equation

An equation is said to be homogeneous if every term in the equation is of the same degree.

A second degree homogeneous equation is of the form $ax^2 + 2hxy + by^2 = 0$

The solution of simultaneous equations gives the point(s) of intersection of their graphs.



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# 5 CHAPTER PARTIAL FRACTIONS

## Rational Fraction

The quotient of two polynomials $\frac{P(x)}{Q(x)}, Q(x) \neq 0$ with no common factors, is called a Rational Fraction. For example, $\frac{x^3 - x^2 + x + 1}{x^2 + 5}$

## Types of Rational Fractions

### (i) Proper Rational Fraction

The rational fraction $\frac{P(x)}{Q(x)}$ is called Proper Rational Fraction if the degree of $P(x)$ is less than the degree of $Q(x)$. For example, $\frac{2}{x-1}, \frac{2x}{x^2+1}, \frac{7x^2}{x^3-3}$

### (ii) Improper Rational Fraction

The rational fraction $\frac{P(x)}{Q(x)}$ is called Improper Rational Fraction if the degree of $P(x)$ is equal to or greater than the degree of $Q(x)$.

For example, $\frac{x}{3x+2}, \frac{x^2+1}{4x+1}$

> Every improper rational fraction can be changed into mixed form by using long division.

## Equal Polynomials

If two polynomials are equal for all values of the variable, then the polynomials have same degree and the coefficients of like powers of the variable in both the polynomials must be equal.

For example, If $tx^3 - sx^2 + 2ax + p = 2x^3 + 3x^2 - 4x + 7$ then $t = 2,\ s = -3,\ a = -2,\ p = 7$

## Partial Fractions

When a single rational fraction is resolved into two or more rational fractions then these resolved rational fractions are called Partial Fractions.

## Partial Fraction Resolution

Expressing a rational fraction as a sum of partial fractions is called Partial Fraction Resolution. For example $\frac{7x+25}{(x-3)(x-4)} = \frac{A}{x-3} + \frac{B}{x-4}$



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## Difference between Conditional Equation and Identity

| Conditional Equation | Identity |
|---|---|
| A type of equation in which two algebraic expressions are equal for particular value(s) of the variable(s).<br>(i) $4x=5$<br>(ii) $2x^2+3x+4=0$ | A type of equation which holds for all values of the variable involved.<br>(i) $(a-b)x=ax-bx$<br>(ii) $(x+2)(x-2)=x^2-4$ |

## Resolution of a Rational Fraction into Partial Fractions

Following are the main points of resolving a rational fraction $\frac{P(x)}{Q(x)}$ into partial fractions:

1. The degree of $P(x)$ must be less than that of $Q(x)$. If not, divide and work with remainder theorem.
2. Clear the given equation of fractions.
3. Equate the coefficients of like terms (powers of $x$).
4. Solve the resulting equations for the coefficients.

Here we shall discuss the methods to break up a rational fraction into partial fractions.

**Case – I** **Resolution of $\frac{P(x)}{Q(x)}$ into partial fractions when $Q(x)$ has only non-repeated linear factors (i.e. denominator contains factors real, linear but not repeated)**

If $Q(x)=(x-a_1)(x-a_2)....(x-a_n)$ where $a_1\neq a_2\neq .....\neq a_n, n\in\mathbb{N}$

$\therefore \frac{P(x)}{Q(x)}=\frac{A_1}{x-a_1}+\frac{A_2}{x-a_2}+....+\frac{A_n}{x-a_n}$ is an identity.

Where the coefficients $A_1, A_2, .....A_n$ are the numbers to be found.

When a rational fraction is resolved into partial fractions, the resultant equation is an identity.

**Case – II** **When $Q(x)$ has repeated linear factors. (i.e. denominator contains factors real, linear but repeated)**

To each linear factor of the form $(x-a)^n, n\geq 2$ and $n$ is a positive integer, there corresponds a sum of $n$ partial fractions as follows:

If $Q(x)=(x-a)^n, n\geq 2,\ n\in\mathbb{N}$



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$$\therefore \frac{P(x)}{Q(x)} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + .... + \frac{A_n}{(x-a)^n} \text{ is an identity.}$$

Where the coefficients $A_1, A_2, ....., A_n$ are the numbers to be found.

**Case – III** **When $Q(x)$ has non-repeated irreducible quadratic factors (i.e. denominator contains factors, real quadratic, but not repeated)**

If the polynomial $Q(x)$ contains non-repeated irreducible quadratic factor then $\frac{P(x)}{Q(x)}$ may be written in the form of partial fractions as; $\frac{Ax+B}{ax^2+bx+c}$, where $A$ and $B$ are the numbers to be found.

A quadratic factor is irreducible if it cannot be written as the product of two linear factors with real co-efficients e.g. $x^2+x+1, x^2+3$

**Case – IV** **When $Q(x)$ has repeated irreducible quadratic factors (i.e. denominator contains factors, real quadratic & repeated).**

If the polynomial $Q(x)$ contains repeated irreducible quadratic factors of the form $(ax^2+bx+c)^n, n \geq 2, n \in \mathbb{N}$

$$\therefore \frac{P(x)}{Q(x)} = \frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + .... + \frac{A_nx+B_n}{(ax^2+bx+c)^n} \text{ is an identity.}$$

Where $A_1, B_1, A_2, B_2, ..., A_n, B_n$ are numbers to be found.

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# 6 CHAPTER SEQUENCE AND SERIES

## SEQUENCE

- A Sequence is an arrangement of numbers subject to some definite rule.
- A Sequence is a function whose domain is a subset of the set of Natural Numbers and range is Real or Complex Numbers. $f : N \to \mathbb{R} / C$

**Key Facts:**

- Notation used for sequence is $\{a_n\}$.
- Each number in the sequence is called a **Term**.
- The $n^{th}$ term or general term of a sequence is denoted by $a_n$.
- $a_1, a_2, a_3, a_4$ be the first, second, third and, fourth terms of a sequence respectively.
- If all members of a sequence are real numbers, then it is called a **Real Sequence.**
- If the domain of a sequence is a finite set then the sequence is called a **Finite Sequence** otherwise, an **Infinite Sequence.**
- An infinite sequence has no last term.

## Arithmetic Sequence or Arithmetic Progression (A.P.)

- A sequence in which every term after the first is obtained from the preceding term by adding a fixed number in it is called an **Arithmetic Sequence** or **Arithmetic Progression.**
- The difference of two consecutive terms of an A.P. is called the **Common Difference.** Symbolically it is written as $a_n - a_{n-1} = d$ where $n > 1$.

d can be any number

## General Term or $n^{th}$ Term of an A.P.

- The general form of an *A.P.* is $a_1, a_1 + d, a_1 + 2d, \cdots, a_1 + (n-1)d$
- The **General** or $n^{th}$ term of an A.P. is written as $a_n = a_1 + (n-1)d$,

  Where, $a_1$ = First term , $n$ = Number of terms

  $d$ = Common Difference, $a_n$ = $n^{th}$ term or general term

The value of n is always a positive integer.

## Arithmetic Mean (A.M.)

- If three terms are in A.P. then middle term is known as **Arithmetic Mean** (A.M.) of the other two terms.
- If $a, A, b$ are in A.P. then $A = \frac{a+b}{2}$
- If $\alpha, \beta, \gamma$ are in A.P. then their A.M. is $\beta$.

## $n$ Arithmetic Means Between Two Numbers

- If $a, A_1, A_2, A_3, \ldots, A_n, b$ are in A.P then the numbers $A_1, A_2, A_3, \ldots, A_n$ are called $n$ **Arithmetic Means** between '$a$' and '$b$'.
- $n^{th}$ arithmetic mean between $a$ and $b$ is given by, $A_n = \frac{a+nb}{n+1}$



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- The sum of $n$ A.M's between any two numbers $a$ and $b$ is equal to $n$-times their A.M

  i.e., $A_1 + A_2 + A_3 + \cdots + A_n = n\left(\frac{a+b}{2}\right)$.

## Series

The sum of an indicated number of terms of a sequence is called a **Series.**

## Arithmetic Series

If $a_1, a_1+d, a_1+2d, a_1+3d, \ldots, a_1+(n-1)d$ are in A.P, then series

$a_1+(a_1+d)+(a_1+2d)+\ldots+\left[a_1+(n-1)d\right]$ is called an **Arithmetic Series.**

## Sum of First '*n*' terms of an Arithmetic Series

- If $a_1$=**first term**, and $a_n = \ell =$ **general term** is given then use formula, $S_n = \frac{n}{2}(a_1+a_n)$,

  $S_n = \frac{n}{2}(a_1+l)$

- The sum of first $n$ terms of arithmetic series can be calculated by using the formula

  $S_n = \frac{n}{2}\{2a_1+(n-1)d\}$.

- If $d=0$ then the above Series becomes $a_1+a_1+a_1+\ldots+n$ terms and $S_n = na_1$

## Summary of Formulae in A.P.

| ARITHMETIC PROGRESSION | ARITHMETIC MEAN | ARITHMETIC SERIES |
|---|---|---|
| 1. $a_n = a_1+(n-1)d$ | 1. $A = \frac{a+b}{2}$ | 1. $S_n = \frac{n}{2}\left[2a_1+(n-1)d\right]$ |
| 2. $d = a_n - a_{n-1}$ , $n>1$ | 2. $A_m = a+\frac{m(b-a)}{n+1}$ | 2. $S_n = \frac{n}{2}(a_1+a_n)$ |
| 3. $n = \frac{a_n-a_1}{d}+1$ | 3. $A_1+A_2+A_3+\ldots+A_n = n\left(\frac{a+b}{2}\right)$ | 3. $S_n = \frac{n}{2}(a_1+l)$ |

## Geometric Sequence or Geometric Progression (G.P.)

- A sequence in which every term after the first is obtained from the preceding term by multiplying it with a non-zero constant number is called a **Geometric Sequence** or **Geometric Progression.**
- The constant ratio of any two consecutive terms of a G.P. is called its **Common Ratio.** Symbolically it is written as

  $r = \frac{a_n}{a_{n-1}}$, where $n>1$.

No term of a G.P can be zero.

## General Term Or $n^{th}$ Term of G.P.

- The general form a G.P. is $a_1, a_1r, a_1r^2, a_1r^3, \cdots, a_1r^{n-1}$
- The **General Term** or $n^{th}$ term of G.P. is $a_n = a_1r^{n-1}$, $(a_n \neq 0)$

  Where $a_1$ = First term, $r$ = **Common Ratio** $a_n$ = General or $n^{th}$ term.

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## Geometric Mean (G.M.)

- If three terms are in G.P, then middle term is known as **Geometric Mean** (G.M.) of the other two terms.
- If $a$, $G$, $b$ are in G.P. then $G = \pm\sqrt{ab}$ .

## $n$ Geometric Means Between Two Numbers

- If $a, G_1, G_2, G_3, ..., G_n, b$ are in G.P. then the numbers $G_1, G_2, G_3, ..., G_n$ are called $n$ geometric means between '$a$' and '$b$'.
- $n^{th}$ G.M. between $a$ and $b$ is given by; $G_n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$
- The product of $n$ G.M's between $a$ and $b$ is $n^{th}$ power of single G.M between $a$ and $b$

  i.e., $G_1.G_2.G_3...G_n = \left(\sqrt{ab}\right)^n$

## Geometric Series

If $a_1, a_1r, a_1r^2, a_1r^3, \cdots, a_1r^{n-1}$ are in G.P. then the series

$a_1 + a_1r + a_1r^2 + a_1r^3 + \cdots + a_1r^{n-1}$ is called **Geometric series.**

## Sum of First '$n$' Terms of a Geometric Series

(a) $S_n = \frac{a_1(1-r^n)}{1-r}$, $|r|<1$.

(b) $S_n = \frac{a_1 - lr}{1-r}$, $|r|<1$

(c) $S_n = \frac{a_1(r^n-1)}{r-1}$, $|r|>1$

(d) $S_n = \frac{lr - a_1}{r-1}$, $|r|>1$

## The Infinite Geometric Series

The series $a_1 + a_1r + a_1r^2 + a_1r^3 + \cdots + a_1r^{n-1} + \cdots\infty$ is called **Infinite Geometric Series.**

## Sum of an Infinite Geometric Series

Sum of infinite geometric series can be calculated by the formula $S_\infty = \frac{a_1}{1-r}, |r|<1$

## Convergent Series

An infinite series is said to be convergent if $n^{th}$ term of series tends to a definite number L as $n \to \infty$

i.e. $\lim_{n\to\infty} a_n = L$



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## Divergent Series

An infinite series is said to be divergent if n$^{th}$ term of series tends to $\pm\infty$ as $n \to \infty$

$$\text{i.e. } \lim_{n\to\infty} a_n = \pm\infty$$

- If $r = 1$, then the series becomes $a_1 + a_1 + a_1 + a_1 + \ldots$ . In this case $S_n$ does not tend to a limit when $n \to \infty$ and the series does not converge.
- If $r = -1$, then the series becomes $a_1 - a_1 + a_1 - a_1 + a_1 - a_1 + \ldots$ and $S_n = \dfrac{a_1 - (-1)^n a_1}{2}$

  i.e., **(a)** $S_n = a_1$ if $n$ is positive odd integer.

  **(b)** $S_n = 0$ if $n$ is positive even integer.

Thus $S_n$ does not tend to a definite number when $n \to \infty$. In such a case we say that series is **Oscillatory.**

## Summary of Formulae in G.P.

| GEOMETRIC PROGRESSION | GEOMETRIC MEAN | GEOMETRIC SERIES |
|---|---|---|
| 1. $a_n = a_1 r^{n-1}$ | 1. $G = \pm\sqrt{ab}$ | For Finite Series: |
| 2. $r = \dfrac{a_n}{a_{n-1}}, \quad r > 1$ | 2. $G_n = a\left(\dfrac{b}{a}\right)^{\frac{n}{n+1}}$ | 1. $S_n = \dfrac{a_1(1-r^n)}{1-r}, \quad \lvert r\rvert < 1$ |
| | 3. $G_1.G_2.G_3....G_n = \left(\sqrt{ab}\right)^n$ | 2. $S_n = \dfrac{a_1(r^n-1)}{r-1}, \quad \lvert r\rvert > 1$ |
| | | For Infinite Series: |
| | | 3. $S_\infty = \dfrac{a_1}{1-r}$ |

## Harmonic Progression (H.P.)

- A sequence of numbers is called a **Harmonic Sequence** or **Harmonic Progression** if the reciprocals of its terms are in Arithmetic Progression.
- General term of H.P is $a_n = \dfrac{1}{a_1 + (n-1)d}$

## Harmonic Mean (H.M.)

- If three terms are in H.P. then the middle term is called **Harmonic Mean** (H.M.) of the other two terms.
- If $a$, $H$, $b$ are in H.P. then $H = \dfrac{2ab}{a+b}$

## *n* Harmonic Means Between Two Numbers

- If $a$, $H_1, H_2, H_3 \ldots, H_n, b$ are in H.P then the numbers $H_1, H_2, H_3 \ldots, H_n$ are called $n$ **Harmonic Means** between $a$ and b.
- $n^{th}$ harmonic mean between $a$ and b is given by, $H.M. = \dfrac{(n+1)ab}{na+b}$.



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## Relations between A.M, G.M and H.M

- $G^2 = A.H$
- $A > G > H$, if $a$ and $b$ are two distinct positive real numbers and $G > 0$
- $A < G < H$, if $a$ and $b$ are two distinct negative real numbers and $G < 0$.
- If $a = b$ then $A = G = H$

- No term of H.P. can be zero.
- Every H.P has a corresponding A.P.
- There is no general method to find the sum of H.P.

**Note:**
A,G,H are in G.P.

## Sigma Notation or Summation Notation

Sum of $1^{st}$ $n$ natural numbers

- $1+2+3+\cdots+n=\sum_{k=1}^{n}k=\frac{n(n+1)}{2}$

Sum of squares of $1^{st}$ $n$ natural numbers

- $1^2+2^2+3^2+\cdots+n^2=\sum_{k=1}^{n}k^2=\frac{n(n+1)(2n+1)}{6}$

Sum of cubes of $1^{st}$ $n$ natural numbers

- $1^3+2^3+3^3+\cdots+n^3=\sum_{k=1}^{n}k^3=\left(\frac{n(n+1)}{2}\right)^2=\frac{n^2(n+1)^2}{4}=\left(\sum_{k=1}^{n}k\right)^2$
- $\sum_{k=1}^{n}\left[k^m-(k-1)^m\right]=n^m$
- $1+1+1+\ldots+n \text{ terms}=\sum_{k=1}^{n}1=n$

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# CHAPTER 7 PERMUTATION, COMBINATION AND PROBABILITY

## Factorial Notation

Let $n$ be a positive integer. Then the product $n(n-1)(n-2)...3.2.1$ is called $n$ factorial.

In above case $n$ factorial is denoted by $n!$ and it can also be written as $\lfloor\underline{n}$ .

**Values to Remember:**

$0! = 1$ (by definition)

$1! = 1$

$2! = 2$

$3! = 6$

$4! = 24$

$5! = 120$

$6! = 720$

$7! = 5040$

$8! = 40320$

$9! = 362880$

$10! = 3628800$

**Caution**

- $(a \pm b)! \neq a! \pm b!$
- $(a.b)! \neq a!.b!$
- $(2n)! \neq 2(n)!$

## Principles of Counting

**(i) Fundamental Principle of Addition**

If an operation can be performed in $m$ ways and another operation (independent of first operation) can be performed in $n$ ways, then either of the two operations can be performed in $m + n$ ways. This is called fundamental principle of addition.

**(ii) Fundamental Principle of Multiplication**

If an operation can be performed in $m$ ways and afterwards a second operation can be performed in $n$ ways then both these operations can be performed in $m \times n$ ways. This is called fundamental principle of multiplication.

**Remember:**

These principles can be extended up to finite number of operations.

## Permutations

The different arrangements which can be made out of given numbers, taken one or more at a time, are called permutations.

OR

An arrangement of a finite number of objects taken one or more at a time is called a permutation of these objects (w.r.t. their order).

**Do you know?**

Permutation and Combination are special cases of fundamental principle of counting.



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**Key Results of Permutation:**

| | |
|---|---|
| (i) | The number of permutations of $n$-distinct objects taken all at a time is $n!$ |
| (ii) | The number of permutations of "$n$" distinct objects taken "$r$" at a time is given by: $^nP_r = \frac{n!}{(n-r)!}\ 0 \le r \le n$ |
| (iii) | $^nP_n = n!$ |
| (iv) | $^nP_0 = 1$ |
| (v) | If out of "$n$" given objects "$p$" are alike of one kind, "$q$" are alike of second kind, "$r$" are alike of third kind and so on then the number of permutations of these $n$-objects taken all at a time is given by: $\frac{n!}{p!\,q!\,r!}$. The formula can also be written as, $\binom{n}{p\ \ q\ \ r}$ |
| (vi) | The number of circular permutations of "$n$" different objects taken all at a time is $(n-1)!$ if clockwise and anti-clockwise orders are different. |
| (vii) | Number of arrangements of $n$-beads (keys) etc, can be arranged on a circular wire (key ring) is $\frac{(n-1)!}{2}$ if clockwise and anti-clockwise orders are not different. |

## Combinations

The different groups or selections which can be made out of a given number of objects, taken one or more at a time (irrespective of their arrangements) are called combinations.

**Key Results of Combinations:**

| | |
|---|---|
| (i) | The number of combinations of "$n$" distinct objects, taken all at a time is $^nC_n = 1$ |
| (ii) | The number of combinations of "$n$" distinct objects taken "$r$" at a time is given by: $^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}\qquad 0 \le r \le n$ |
| (iii) | The number of combinations of "$n$" distinct objects taken "$r$" at a time when "$p$" particular objects are always included is given by: $^{n-p}C_{r-p}$ |
| (iv) | The number of combinations of "$n$" distinct objects taken "$r$" at a time when "$p$" particular objects are never included is given by: $^{n-p}C_r$ |
| (v) | $^nC_0 = {}^nC_n = 1$ |
| (vi) | $^nC_r = {}^nC_{n-r}$ Which is called **Complementary Combination.** |
| (vii) | $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$ |
| (viii) | If $^nC_r = {}^nC_s$, then either $r = s$ or $r + s = n$ |
| (ix) | $\frac{^nC_r}{^nC_{r-1}} = \frac{n-r+1}{r}$ |
| (x) | $^nC_r.r! = {}^nP_r$ |
| (xi) | $^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n = 2^n$ |
| (xii) | The number of ways of selecting one or more objects out of "$n$" objects is $2^n - 1$ |



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**Note:**

In most cases, given problem cannot be determined whether it belongs to permutation or combination. On the basis of few words or on the assumption in expression of language given in the problem, we can identify that problem according to the following table.

**Can you Guess?**

Number of combinations of $n$ distinct objects taken $r$ at a time when $p$ objects always included and $q$ objects never included.

| PERMUTATION | COMBINATION |
|---|---|
| Arrangements,<br>Standing or sitting in a row or in a circle<br>Problem regarding digits, letters(A,B,C….etc)<br>Formation of words, numbers etc | Selection,Choice,Draw ,etc<br>Distributation, formation of a group, committee, team etc<br>Problem regarding geometry |

## Terms Related to Probability

| | |
|---|---|
| **Random Experiment** | An experiment in which all possible outcomes are known in advance. |
| **Sample Space** | The set of all possible outcomes of an experiment is called the sample space. It is denoted by $S$. |
| **Event** | Any subset of the sample space is called an event. |
| **Mutually Exclusive Events or Disjoint Events** | Two events A and B are said to be mutually exclusive or disjoint if and only if they cannot both occur at the same time i.e. they have no point in common.<br>$A \cap B = \phi$ |
| **Not Mutually Exclusive Events or Overlapping Events.** | Two events A and B are said to be not mutually exclusive or overlapping if and only if they have at least one point in common $A \cap B \neq \phi$ |
| **Equally Likely Events** | Two events A and B are said to be equally likely when one event is as likely to occur as the other. |
| **Independent Events/Dependent Events** | Two or more events such that occurrence of any one does not depend on the occurrence of other, they are said to be independent events. Otherwise dependent events. |

## Probability

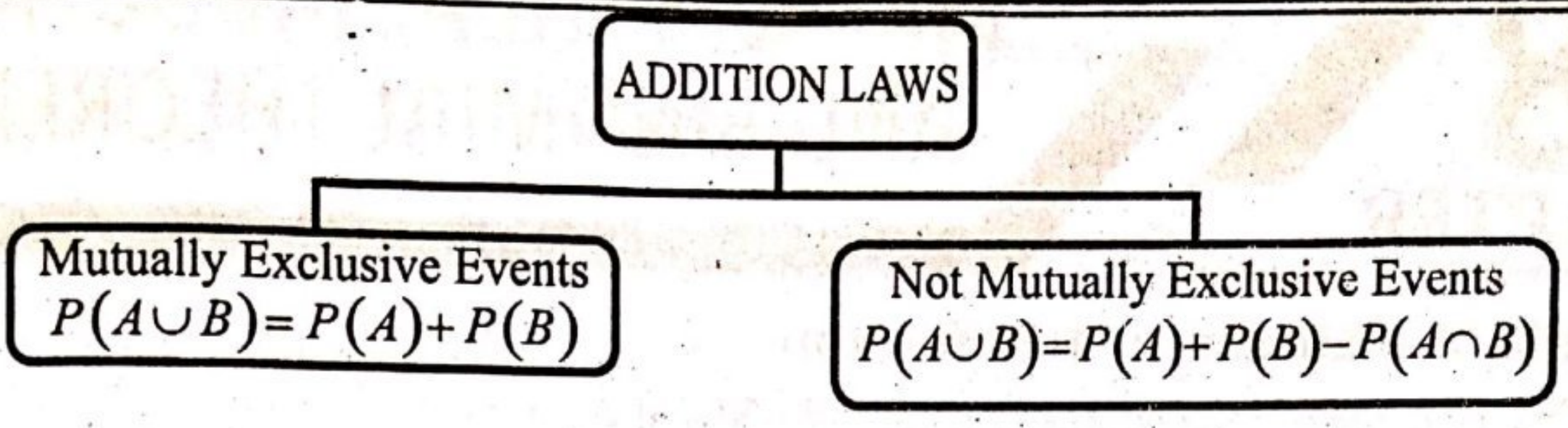
Probability is the numerical evaluation of a chance that a particular event would occur.

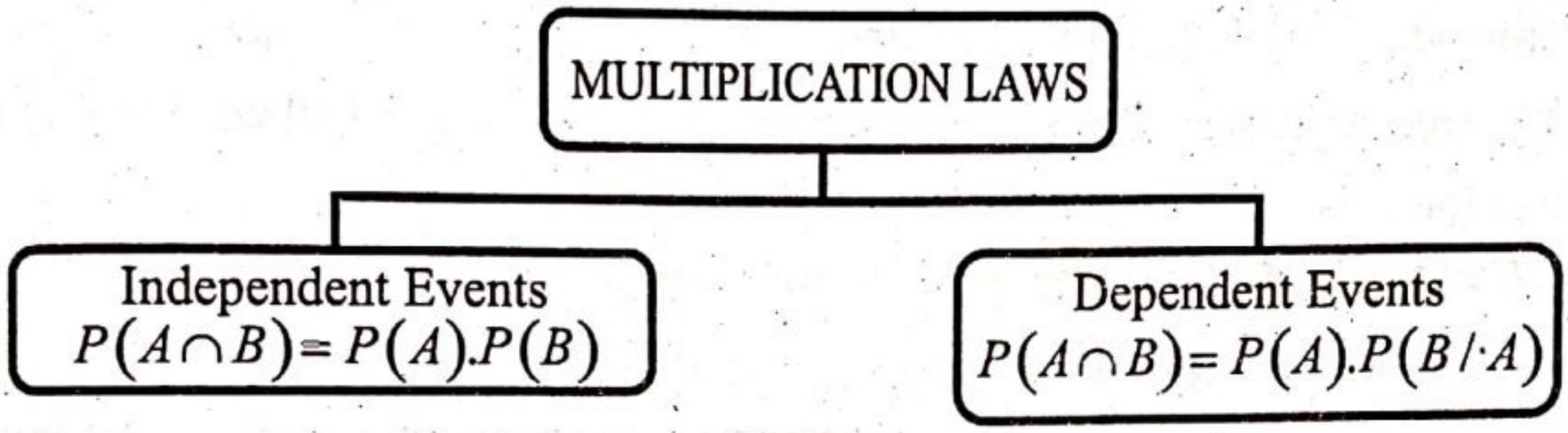
**Results:**

- Probability of occurrence of an event $E$: $P(E) = \dfrac{n(E)}{n(S)}$
- Probability of not happening of an event: $P(\bar{E}) = 1 - P(E)$
- $0 \leq P(E) \leq 1$
- $P(E) + P(\bar{E}) = 1$
- If E is an Impossible Event then $P(E) = 0$
- If E is a Certain Event then $P(E) = 1$



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**Remember:**

- Independent events are always taken from different experiments while mutually exclusive events are taken from a single experiment.
- Independent events can happen together while mutually exclusive events cannot happen together.
- Independent events are connected by the word "AND" but mutually exclusive events are connected by the word "OR".

**TIP:**
Generally in probability "AND" means multiplication and "OR" means addition.

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# 8 CHAPTER MATHEMATICAL INDUCTION AND BINOMIAL THEOREM

## Principle of Mathematical Induction

The principle of mathematical induction is stated as follows:

If a proposition or statement $S(n)$ for each positive integer $n$ is such that

(i) $S(1)$ is true i.e., $S(n)$ is true for $n=1$ and

(ii) $S(k+1)$ is true whenever $S(k)$ is true for any positive integer $k$, then $S(n)$ is true for all positive integers.

## Principle of Extended Mathematical Induction

Let $i$ be an integer. If a formula or statement $S(n)$ for $n \geq i$ is such that

(i) $S(i)$ is true and

(ii) $S(k+1)$ is true whenever $S(k)$ is true for any integer $n \geq i$.

Then $S(n)$ is true for all integers $n \geq i$.

There is no integer n for which $3^n$ is even

## Binomial Expression

An algebraic expression consisting of two terms such as $a+x$, $x-2y$, $ax+b$ etc is called a **Binomial** or **Binomial Expression**.

## Binomial Theorem

The following expansions are valid for any $n \in Z^+$. ($n$ is called index)

(i) $(a+b)^n = a^n + \frac{n}{1!}a^{n-1}b^1 + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \ldots + b^n$

(ii) $(a+b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b^1 + {}^nC_2a^{n-2}b^2 + \ldots + {}^nC_ra^{n-r}b^r + \ldots + {}^nC_nb^n$

(iii) $(a+b)^n = \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r$, $n \in N$

Where $a$ and $b$ are real numbers.

## Properties of the Binomial Expansion

The following points can be observed in the expansion of $(a+b)^n$, $n \in N$.

(i) The number of terms in the expansion is one greater than its index. (i.e. number of terms = $n+1$)

(ii) The sum of exponents of $a$ and $b$ in each term of the expansion is equal to its index.

(iii) The exponent of $a$ decreases from index to zero.

(iv) The exponent of $b$ increases from zero to index.

(v) The coefficients of the terms equidistant from beginning and end of the expansion are equal as ${}^nC_r = {}^nC_{n-r}$.

(vi) The $(r+1)^{th}$ or general term in the expansion of $(a+b)^n$ is given by $T_{r+1} = \binom{n}{r}a^{n-r}b^r$



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(vii) Middle term(s) in the expansion of $(a+b)^n$

- If $n$ is even, then there is only one middle term *i.e.* $\left(\frac{n+2}{2}\right)^{th}$ term.
- If $n$ is odd, then there will be two middle terms. *i.e.* $\left(\frac{n+1}{2}\right)^{th}$ term and $\left(\frac{n+3}{2}\right)^{th}$ term.

(viii) ${}^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_n$ are called **Binomial Coefficients.**

(ix) Sum of binomial coefficients is $2^n$ *i.e.*, ${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots\ldots + {}^nC_n = 2^n$

(x) Sum of even binomial coefficients = Sum of odd Binomial Coefficients $= 2^{n-1}$
i.e., ${}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots\ldots + {}^nC_n = {}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots\ldots + {}^nC_{n-1} = 2^{n-1}$

## The Binomial Theorem When the Index *n* is a Negative Integer Or a Fraction

(i) When $n$ is negative integer or a fraction, and $|x|<1 (i.e., -1<x<1)$, then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r + \ldots$$

is called binomial series.

(ii) The general term in the expansion is $T_{r+1} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r$

(iii) The number of terms in the expansion are always infinite.

## Some Particular Cases of the Expansion of $(1+x)^n$, When $n<0$

| BINOMIAL EXPANSION | GENERAL TERM OR $(r+1)^{th}$ TERM |
|---|---|
| $(1+x)^{-1}$ | $(-1)^r x^r$ |
| $(1+x)^{-2}$ | $(-1)^r (r+1)x^r$ |
| $(1+x)^{-3}$ | $(-1)^r \frac{(r+1)(r+2)}{2}x^r$ |
| $(1-x)^{-1}$ | $x^r$ |
| $(1-x)^{-2}$ | $(r+1)x^r$ |
| $(1-x)^{-3}$ | $\frac{(r+1)(r+2)}{2}x^r$ |



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# CHAPTER 9 FUNDAMENTALS OF TRIGONOMETRY

## Trigonometry

The branch of Mathematics, which deals with the study of measurements of triangles called **trigonometry**.

## Types of Trigonometry

There are two types of trigonometry.

(i) Plane Trigonometry

(ii) Spherical Trigonometry

### (i) Plane Trigonometry

In this type we are concerned with angles, triangles and other figures, which lie in plane.

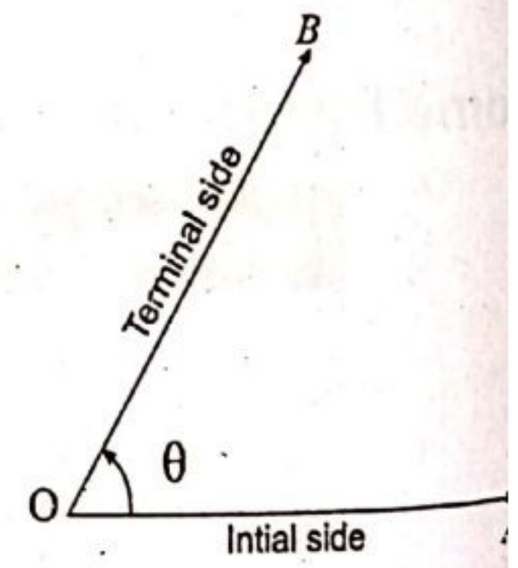
### (ii) Spherical Trigonometry

It is the branch of trigonometry concerned with the measurement of the angles and sid of spherical triangles. Spherical triangles are the triangles which lie on a sphere and sid of these triangles are circular arcs.

## Angle

Two rays with a common starting point (vertex) form an angle. One of the rays of angle is called **initial side** and the other as **terminal side**. Showing the direction of rotation from the initial side to the terminal side identifies the angle.

From figure, $m\angle AOB = \theta$




**OR**

In elementary geometry an angle is defined as the union of two non-collinear rays, whic have a common end point.

- An angle is said to be positive/negative if the rotation is anti-clock wise/clock wise.

## Systems of Measures of Angles

There are two commonly used systems of units for measurement of angles in trigonometry.

(i) Sexagesimal System (English System)

(ii) Circular System

### (i) Sexagesimal System

In this system of measurement of angles, the angles are measured in degrees, minutes an seconds.

## Degree

If the circumference of a circle is divided into 360 congruent (equal in length) parts, th angle subtended by one part at the center of the circle is called a degree.



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**Important Results to Remember:**

| | |
|---|---|
| One degree $=1^\circ=60'$ | One minute $=1'=\left(\frac{1}{60}\right)^\circ$ |
| One minute $=1'=60''$ | One second $=1''=\left(\frac{1}{60}\right)'$ |
| One degree $=1^\circ=3600''$ | One second $=1''=\left(\frac{1}{3600}\right)^\circ$ |

- As this system of measurement of angles owes its origin to the English and because 90, 60 are multiples of 6 and 10, so it is known as **English system** or **Sexagesimal system.**

**(ii) Circular System**

There is another system of angular measurement, called the circular system. In this system, the unit of measurement of angles is the ''radian''.

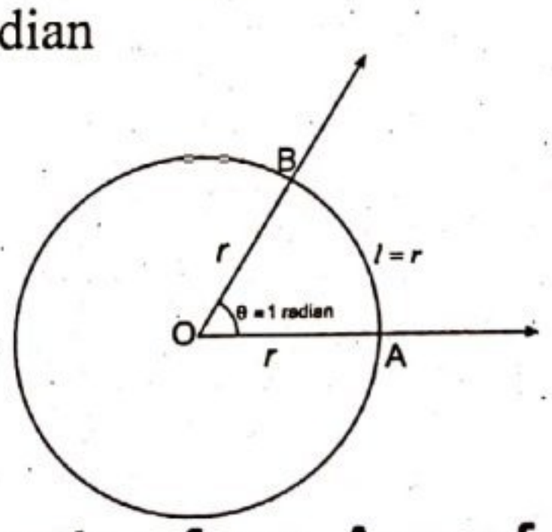
**Remember that:**

- one rotation $=360^\circ$
- $\frac{1}{2}$ rotations $=180^\circ$ (Straight Angle)
- $\frac{1}{4}$ rotations $=90^\circ$ (Right Angle)

## Radian

Radian is the measure of the angle subtended at the center of the circle by an arc, whose length is equal to the radius of the circle.

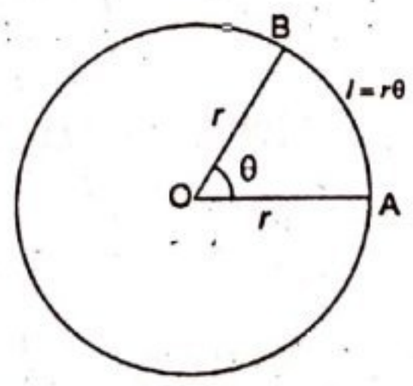
From figure, $m\angle AOB=1$ radian




## Relation between the Length of an Arc of a Circle and the Circular Measure of its Central Angle

Let $r$ be the radius, $l$ is the length of arc and $\theta$ is the circular measure of the central angle of the circle. Then,

- $l=r\theta$ or $\theta=\frac{l}{r}$, if $\theta$ is in radians
- $l=r\frac{\pi}{180}\theta$, If $\theta$ is in degree




**Note:**

Remember that '**r**' and '**ℓ**' are measured in terms of the same unit and the angle $\theta$ is always measured in radians.

## Relationship between Degree and Radians

(i) $1 \text{ radian}=\frac{180^\circ}{\pi}\approx 57.296^\circ$

(ii) $1^\circ=\frac{\pi}{180}\text{radian}\approx 0.0175\text{radian}$



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## Area of sector

- Area of a Sector: $A=\frac{1}{2}r^2\theta$ where $r=$ Radius of circle, $\theta=$ Central angle in radian
- Area of a Sector: $A=\frac{1}{2}rl$ where $r=$ Radius of circle, $l=$ Arc length of circle

## Clock Problems

| Hour Hand | Minute Hand |
|---|---|
| In one minute $=\left(\frac{1}{2}\right)^\circ$ | One minute $=6^\circ$ |
| In one hour $=30^\circ$ | One hour $=360^\circ$ |

## Some Standard Interconvertions between Degrees and Radians

| ANGLE IN DEGREES | ANGLE IN RADIANS | ANGLE IN DEGREES | ANGLE IN RADIANS |
|---|---|---|---|
| $0^\circ$ | $0$ | $210^\circ$ | $\frac{7\pi}{6}$ |
| $30^\circ$ | $\frac{\pi}{6}$ | $225^\circ$ | $\frac{5\pi}{4}$ |
| $45^\circ$ | $\frac{\pi}{4}$ | $240^\circ$ | $\frac{4\pi}{3}$ |
| $60^\circ$ | $\frac{\pi}{3}$ | $270^\circ$ | $\frac{3\pi}{2}$ |
| $90^\circ$ | $\frac{\pi}{2}$ | $300^\circ$ | $\frac{5\pi}{3}$ |
| $120^\circ$ | $\frac{2\pi}{3}$ | $315^\circ$ | $\frac{7\pi}{4}$ |
| $135^\circ$ | $\frac{3\pi}{4}$ | $330^\circ$ | $\frac{11\pi}{6}$ |
| $150^\circ$ | $\frac{5\pi}{6}$ | $360^\circ$ | $2\pi$ |
| $180^\circ$ | $\pi$ | | |

## General Angles (Co-Terminal Angles)

The angles having same initial and terminal sides are known as co-terminal angles.
General angle $=\theta+2k\pi;k\in Z$ all are co-terminal with $\theta$.

## Angle in the Standard Position

An angle is said to be in standard position if its vertex lies at the origin of a rectangular coordinate system and its initial side along the positive $x-$axis .

## Quadrantal Angles

An angle $\theta$ whose terminal side lies on $x-$axis or $y-$axis is called Quadrantal angle.

For example, $0^\circ,90^\circ,180^\circ,\ldots\ldots$ all are Quadrantal angles. In general $\theta=\frac{n\pi}{2};n\in Z$

## Trigonometric Ratios For a Right Triangle

Consider a right triangle $AB$ with $m\angle$ $=90^\circ$ and sides $a,b,c$ as shown in figure.



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Let $m\angle A = \theta$ radians

(i) $\sin\theta = \frac{a}{c} = \frac{\text{Opposite side}}{\text{Hypotenuse}}$

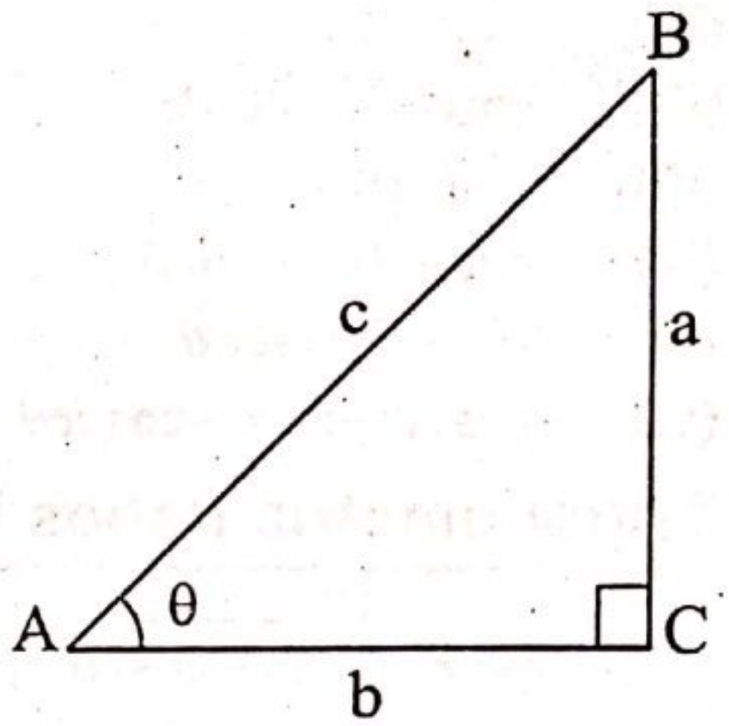
(ii) $\cos\theta = \frac{b}{c} = \frac{\text{Adjacent side}}{\text{Hypotenuse}}$

(iii) $\tan\theta = \frac{a}{b} = \frac{\text{Opposite side}}{\text{Adjacent side}}$

(iv) $\cot\theta = \frac{b}{a} = \frac{\text{Adjacent side}}{\text{Opposite side}}$

(v) $\sec\theta = \frac{c}{b} = \frac{\text{Hypotenuse}}{\text{Adjacent side}}$

(vi) $\cos ec\theta = \frac{c}{a} = \frac{\text{Hypotenuse}}{\text{Opposite side}}$




## Relationship Among Trigonometric Ratios

(i) $\tan\theta = \frac{\sin\theta}{\cos\theta}$

(ii) $\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}$

(iii) $\sec\theta = \frac{1}{\cos\theta}$

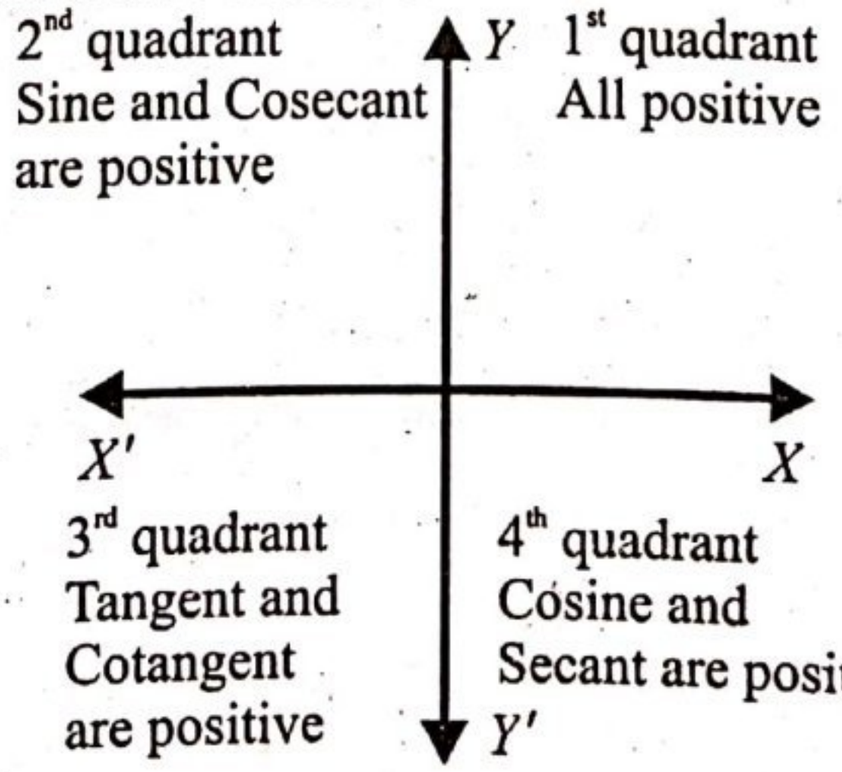
## Fundamental Identities

(i) $\sin^2\theta + \cos^2\theta = 1$

(ii) $1 + \tan^2\theta = \sec^2\theta$

(iii) $\cos ec^2\theta - \cot^2\theta = 1$

## Signs of the Trigonometric Ratios






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## Functions of Negative Angles

(i) $\sin(-\theta) = -\sin\theta$

(ii) $\cos(-\theta) = \cos\theta$

(iii) $\tan(-\theta) = -\tan\theta$

(iv) $\cot(-\theta) = -\cot\theta$

(v) $\sec(-\theta) = \sec\theta$

(vi) $\operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta$

## Trigonometric Ratios in Terms of Other Trigonometric Ratios

| | | | | | |
|---|---|---|---|---|---|
| $\sin\theta$ | $\sqrt{1-\cos^2\theta}$ | $\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$ | $\frac{1}{\sqrt{1+\cot^2\theta}}$ | $\frac{\sqrt{\sec^2\theta-1}}{\sec\theta}$ | $\frac{1}{\operatorname{cosec}\theta}$ |
| $\cos\theta$ | $\sqrt{1-\sin^2\theta}$ | $\frac{1}{\sqrt{1+\tan^2\theta}}$ | $\frac{\cot\theta}{\sqrt{1+\cot^2\theta}}$ | $\frac{1}{\sec\theta}$ | $\frac{\sqrt{\operatorname{cosec}^2\theta-1}}{\operatorname{cosec}\theta}$ |
| $\tan\theta$ | $\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$ | $\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}$ | $\frac{1}{\cot\theta}$ | $\sqrt{\sec^2\theta-1}$ | $\frac{1}{\sqrt{\operatorname{cosec}^2\theta-1}}$ |
| $\cot\theta$. | $\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}$ | $\frac{\cos\theta}{\sqrt{1-\cos^2\theta}}$ | $\frac{1}{\tan\theta}$ | $\frac{1}{\sqrt{\sec^2\theta-1}}$ | $\sqrt{\operatorname{cosec}^2\theta-1}$ |
| $\sec\theta$ | $\frac{1}{\sqrt{1-\sin^2\theta}}$ | $\frac{1}{\cos\theta}$ | $\sqrt{1-\tan^2\theta}$ | $\frac{\sqrt{1+\cot^2\theta}}{\cot\theta}$ | $\frac{\operatorname{cosec}\theta}{\sqrt{\operatorname{cosec}^2\theta-1}}$ |
| $\operatorname{cosec}\theta$ | $\frac{1}{\sin\theta}$ | $\frac{1}{\sqrt{1-\cos^2\theta}}$ | $\frac{\sqrt{1+\tan^2\theta}}{\tan\theta}$ | $\sqrt{1+\cot^2\theta}$ | $\frac{\sec\theta}{\sqrt{\sec^2\theta-1}}$ |



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# 10 CHAPTER TRIGONOMETRIC IDENTITIES, SUM AND DIFFERENCE OF ANGLES

## Distance Formula

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points then the distance "d" between them is defined as

$$d = |PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

## Fundamental Law of Trigonometry

For any two angles $\alpha$ and $\beta$ (real numbers)

$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

Which is called **fundamental law of trigonometry.**

## Deductions from Fundamental Law of Trigonometry

(i) $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

(ii) $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

(iii) $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

(iv) $\tan(\alpha + \beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

(v) $\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

**Remember:**

(i) $\cot(\alpha + \beta) = \dfrac{\cot\alpha\cot\beta - 1}{\cot\beta + \cot\alpha}$

(ii) $\cot(\alpha - \beta) = \dfrac{\cot\alpha\cot\beta + 1}{\cot\beta - \cot\alpha}$

## Results

If $\alpha, \beta, \gamma$ are the angles of $\Delta ABC$, then

(i) $\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma$

(ii) $\tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2} + \tan\dfrac{\beta}{2}\tan\dfrac{\gamma}{2} + \tan\dfrac{\gamma}{2}\tan\dfrac{\alpha}{2} = 1$

(iii) $\cot\alpha\cot\beta + \cot\beta\cot\gamma + \cot\gamma\cot\alpha = 1$

(iv) $\cot\dfrac{\alpha}{2} + \cot\dfrac{\beta}{2} + \cot\dfrac{\gamma}{2} = \cot\dfrac{\alpha}{2}\cot\dfrac{\beta}{2}\cot\dfrac{\gamma}{2}$

## Allied Angles

The angles associated with basic angles of measure $\theta$ to a right angle or its multiple are called allied angles. e.g. $90° \pm \theta$, $180° \pm \theta$, $270° \pm \theta$, $360° \pm \theta$ and In-general $(n \times 90)° \pm \theta$ or $\dfrac{n\pi}{2} \pm \theta; n \in \mathbb{Z}$ all are called allied angles.



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| Sine | Cosine | Tangent |
|---|---|---|
| $\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta$ | $\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta$ | $\tan\left(\frac{\pi}{2}-\theta\right)=\cot\theta$ |
| $\sin\left(\frac{\pi}{2}+\theta\right)=\cos\theta$ | $\cos\left(\frac{\pi}{2}+\theta\right)=-\sin\theta$ | $\tan\left(\frac{\pi}{2}+\theta\right)=-\cot\theta$ |
| $\sin(\pi-\theta)=\sin\theta$ | $\cos(\pi-\theta)=-\cos\theta$ | $\tan(\pi-\theta)=-\tan\theta$ |
| $\sin(\pi+\theta)=-\sin\theta$ | $\cos(\pi+\theta)=-\cos\theta$ | $\tan(\pi+\theta)=\tan\theta$ |
| $\sin\left(\frac{3\pi}{2}-\theta\right)=-\cos\theta$ | $\cos\left(\frac{3\pi}{2}-\theta\right)=-\sin\theta$ | $\tan\left(\frac{3\pi}{2}-\theta\right)=\cot\theta$ |
| $\sin\left(\frac{3\pi}{2}+\theta\right)=-\cos\theta$ | $\cos\left(\frac{3\pi}{2}+\theta\right)=\sin\theta$ | $\tan\left(\frac{3\pi}{2}+\theta\right)=-\cot\theta$ |
| $\sin(2\pi-\theta)=-\sin\theta$ | $\cos(2\pi-\theta)=\cos\theta$ | $\tan(2\pi-\theta)=-\tan\theta$ |
| $\sin(2\pi+\theta)=\sin\theta$ | $\cos(2\pi+\theta)=\cos\theta$ | $\tan(2\pi+\theta)=\tan\theta$ |

## Solving Techniques

**(i)** If $\theta$ is added to or subtracted from odd multiple of right angle (i.e. $(2n+1)\frac{\pi}{2}\pm\theta$) then trigonometric ratios change into co-ratios and vice versa.
i.e $\sin\theta \rightleftarrows \cos\theta$, $\tan\theta \rightleftarrows \cot\theta$, $\sec\theta \rightleftarrows \operatorname{cosec}\theta$
e.g. $\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta, \cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta,$

**(ii)** If $\theta$ is added to or subtracted from even multiple of $\frac{\pi}{2}$ (multiple of $\pi$) then trigonometric ratios shall remain the same.
e.g. $\sec(\pi+\theta)=-\sec\theta$

**(iii)** The sign of the results is concerned with respect to the quadrant in which the terminal arm of the angle lies.

**Results:**

**(i)** $\tan\left(\frac{\pi}{4}+\theta\right)=\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}$

**(ii)** $\tan\left(\frac{\pi}{4}-\theta\right)=\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}$

## Properties of Complementary Angles

Let $\alpha$ and $\beta$ be the complementary angles that is $\alpha+\beta=\frac{\pi}{2}$ then

$\sin\alpha=\sin\left(\frac{\pi}{2}-\beta\right)=\cos\beta$

**Moreover**

**(i)** $\tan\alpha\tan\beta=\cot\alpha\cot\beta=1$

**(ii)** $\frac{\tan\alpha}{\cot\beta}=\frac{\cot\alpha}{\tan\beta}=1$

**(iii)** $\sin^2\alpha+\sin^2\beta=\cos^2\alpha+\cos^2\beta=1$



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So that

$\sin\alpha = \cos\beta$

$\tan\alpha = \cot\beta$

$\sec\alpha = \operatorname{cosec}\beta$

## Important Ratios to Remember

(i) $\sin 15° = \cos 75° = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$

(ii) $\cos 15° = \sin 75° = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$

(iii) $\tan 15° = \cot 75° = \dfrac{\sqrt{3}-1}{\sqrt{3}+1}$

(iv) $\cot 15° = \tan 75° = \dfrac{\sqrt{3}+1}{\sqrt{3}-1}$

(v) $\sec 15° = \operatorname{cosec} 75° = \dfrac{2\sqrt{2}}{\sqrt{3}+1}$

(vi) $\operatorname{cosec} 15° = \sec 75° = \dfrac{2\sqrt{2}}{\sqrt{3}-1}$

## Important Identities

| Double Angle | Half Angle | Triple Angle |
|---|---|---|
| $\sin 2\alpha = 2\sin\alpha\cos\alpha$ | $\sin\dfrac{\alpha}{2} = \pm\sqrt{\dfrac{1-\cos\alpha}{2}}$ | $\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$ |
| $\cos 2\alpha = \begin{cases}\cos^2\alpha - \sin^2\alpha \\ 1-2\sin^2\alpha \\ 2\cos^2\alpha - 1\end{cases}$ | $\cos\dfrac{\alpha}{2} = \pm\sqrt{\dfrac{1+\cos\alpha}{2}}$ | $\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$ |
| $\tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha}$ | $\tan\dfrac{\alpha}{2} = \pm\sqrt{\dfrac{1-\cos\alpha}{1+\cos\alpha}}$ | $\tan 3\alpha = \dfrac{3\tan\alpha - \tan^3\alpha}{1-3\tan^2\alpha}$ |

**Note:**

$\sin 2\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$

$\cos 2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

## Expression of Higher Power of Trigonometric Functions

(i) $\sin^2\alpha = \dfrac{1-\cos 2\alpha}{2}$

(ii) $\cos^2\alpha = \dfrac{1+\cos 2\alpha}{2}$



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(iii) $\sin^3\alpha = \frac{3}{4}\sin\alpha - \frac{1}{4}\sin 3\alpha$

(iv) $\cos^3\alpha = \frac{3}{4}\cos\alpha + \frac{1}{4}\cos 3\alpha$

(v) $\sin^4\alpha = \frac{3}{8} - \frac{1}{2}\cos 2\alpha + \frac{1}{8}\cos 4\alpha$

(vi) $\cos^4\alpha = \frac{3}{8} + \frac{1}{2}\cos 2\alpha + \frac{1}{8}\cos 4\alpha$

## Sum/Differences to Products Formulas

(i) $\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

(ii) $\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$

(iii) $\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

(iv) $\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$

## Products to Sum/Differences Formulas

(i) $2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$

(ii) $2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)$

(iii) $2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$

(iv) $-2\sin\alpha\sin\beta = \cos(\alpha+\beta) - \cos(\alpha-\beta)$

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# 11 CHAPTER

# TRIGONOMETRIC FUNCTIONS AND THEIR GRAPHS

## Domains and Ranges of Trigonometric Functions

| TRIGONOMETRIC FUNCTION | DOMAIN | RANGE |
|---|---|---|
| $y = \sin x$ | $x \in R$ | $-1 \le y \le 1$ |
| $y = \cos x$ | $x \in R$ | $-1 \le y \le 1$ |
| $y = \tan x$ | $x \in R$ But $x \neq (2k+1)\frac{\pi}{2}, k \in Z$ | $y \in R$ |
| $y = \cot x$ | $x \in R$ But $x \neq k\pi, k \in Z$ | $y \in R$ |
| $y = \sec x$ | $x \in R$ But $x \neq (2k+1)\frac{\pi}{2}, k \in Z$ | $y \ge 1$ or $y \le -1$ |
| $y = \operatorname{cosec} x$ | $x \in R$ But $x \neq k\pi, k \in Z$ | $y \ge 1$ or $y \le -1$ |

## Period of Trigonometric Function

Let $f(x)$ be a trigonometric function. Let "$p$" be the smallest positive number such that

$f(x+p) = f(x) \quad \forall x \in R$

Then $p$ is called period of trigonometric function $f(x)$

| Trigonometric Function | $\sin x$ | $\cos x$ | $\sec x$ | $\operatorname{cosec} x$ | $\tan x$ | $\cot x$ |
|---|---|---|---|---|---|---|
| Period | $2\pi$ | $2\pi$ | $2\pi$ | $2\pi$ | $\pi$ | $\pi$ |

## Techniques to Find the Period of a Function

**Note:**
Constant function is periodic with no fundamental period

**(i)** Period of $\sin nx$, $\cos nx$, $\sec nx$, $\operatorname{cosec} nx$ is $\frac{2\pi}{n}$.

**(ii)** Period of $\tan nx$ and $\cot nx$ is $\frac{\pi}{n}$.

**(iii)** If $f(x)$ is periodic with period "$p$" then $\frac{1}{f(x)}$ and $\sqrt{f(x)}$ are also periodic with the same period "$p$".

**(iv)** $|\sin x|, |\cos x|, |\sec x|, |\operatorname{cosec} x|$ are periodic functions with period $\pi$. (i.e. half of the standard period)

**(v)** $|\tan x|, |\cot x|$ are periodic functions with period $\pi$. (i.e. no change in standard period)

**(vi)** $\sin^n x, \cos^n x, \sec^n x, \operatorname{cosec}^n x$ are periodic functions with period $2\pi$ when $n$ is odd and $\pi$ (half of the standard period) when $n$ is even.

**(vii)** $\tan^n x$ and $\cot^n x$ are periodic functions with period $\pi$ for both $n$ even or odd.

**Do You Know:**
The period of $f(x)+g(x)$ if $f(x)$ and $g(x)$ are periodic?



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## Amplitude

The amplitude is the one-half the vertical distance between the lowest and the highest point of the graph.

## Phase Shift

The phase shift is the distance to the right or left that the graph is changed from its normal position.

## Frequency

Frequency is the reciprocal of the period

$$\text{Frequency} = \frac{1}{\text{period}}$$

**NOTE**
The amplitude is associated with only the sine and cosine functions.

## The General Form of a Trigonometric Function

- $y = A.f(Bx + C)$

The following terms are related to the trigonometric functions.

### (1) Domain

The domain of functions is $\frac{1}{B}$ (Domain of $f$)

### (2) Range

The range of the function is $A$ (Range of $f$)

### (3) Amplitude

The amplitude of a function $= |A|$

### (4) Period

Period of function $= \frac{p}{B}$, where $p$ is the normal period of the function.

### (5) Frequency

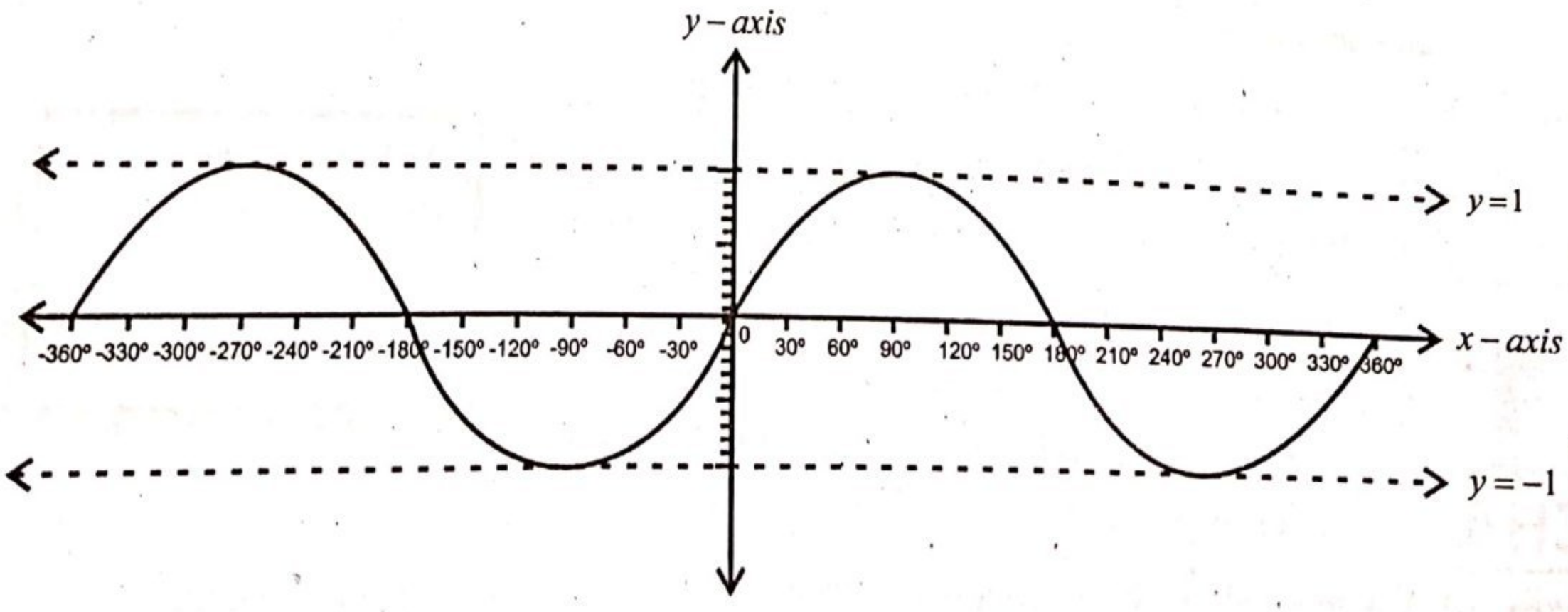
$$\text{Frequency} = \frac{1}{\text{period}}$$

### (6) Phase Shift

The phase shift of a function $= -\frac{C}{B}$

### (7) Graphs of Trigonometric Functions

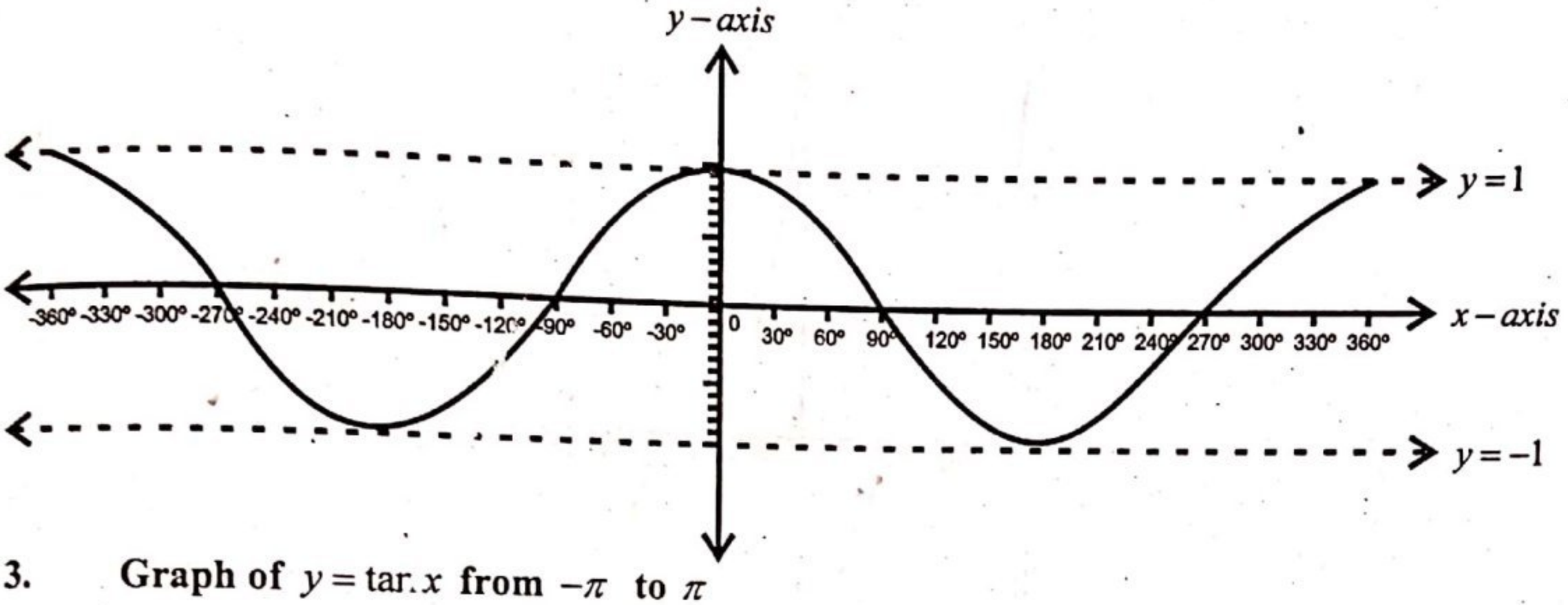
**1. Graph of $y = \sin x$ from $-2\pi$ to $2\pi$**



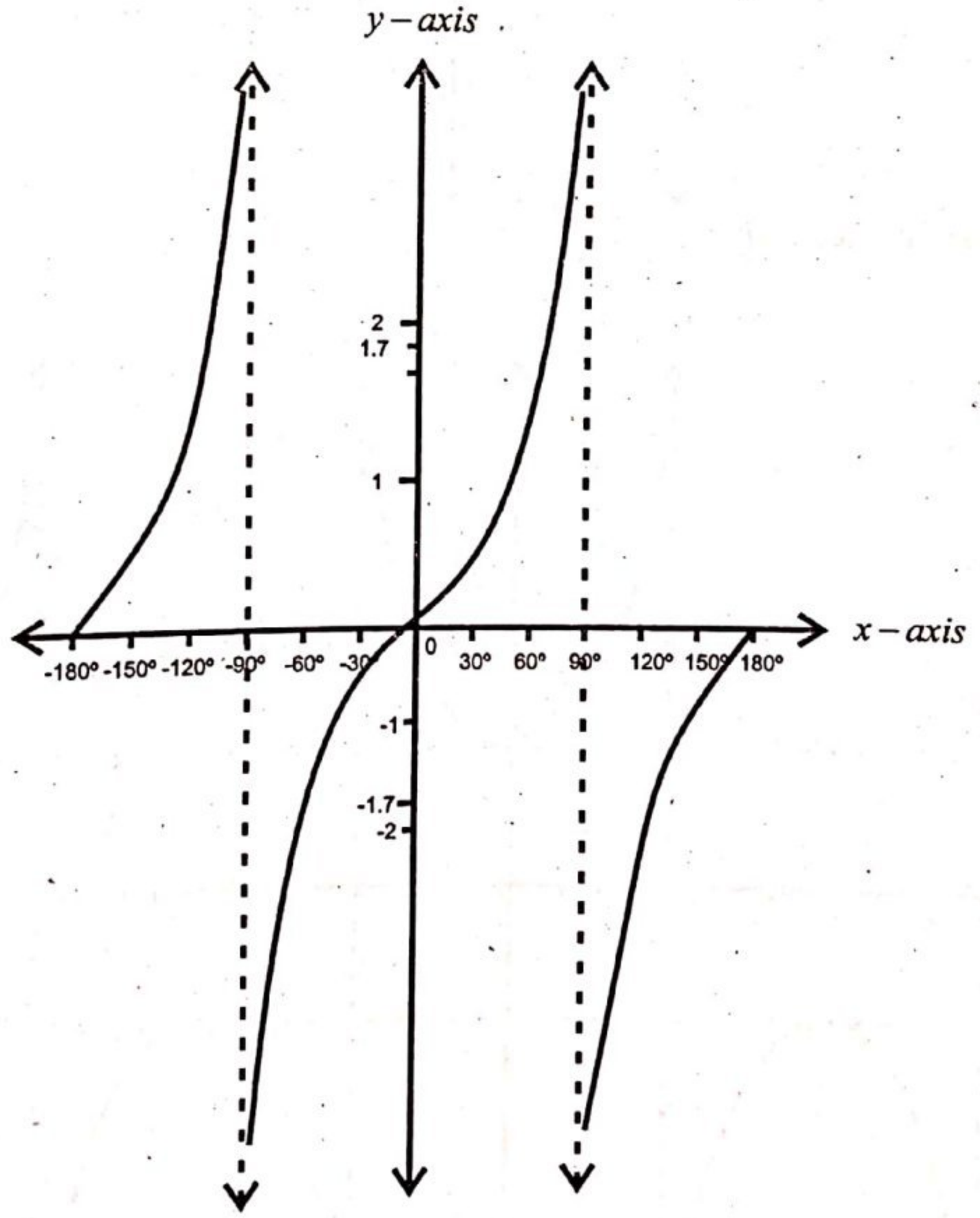



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**2. Graph of $y = \cos x$ from $-2\pi$ to $2\pi$**



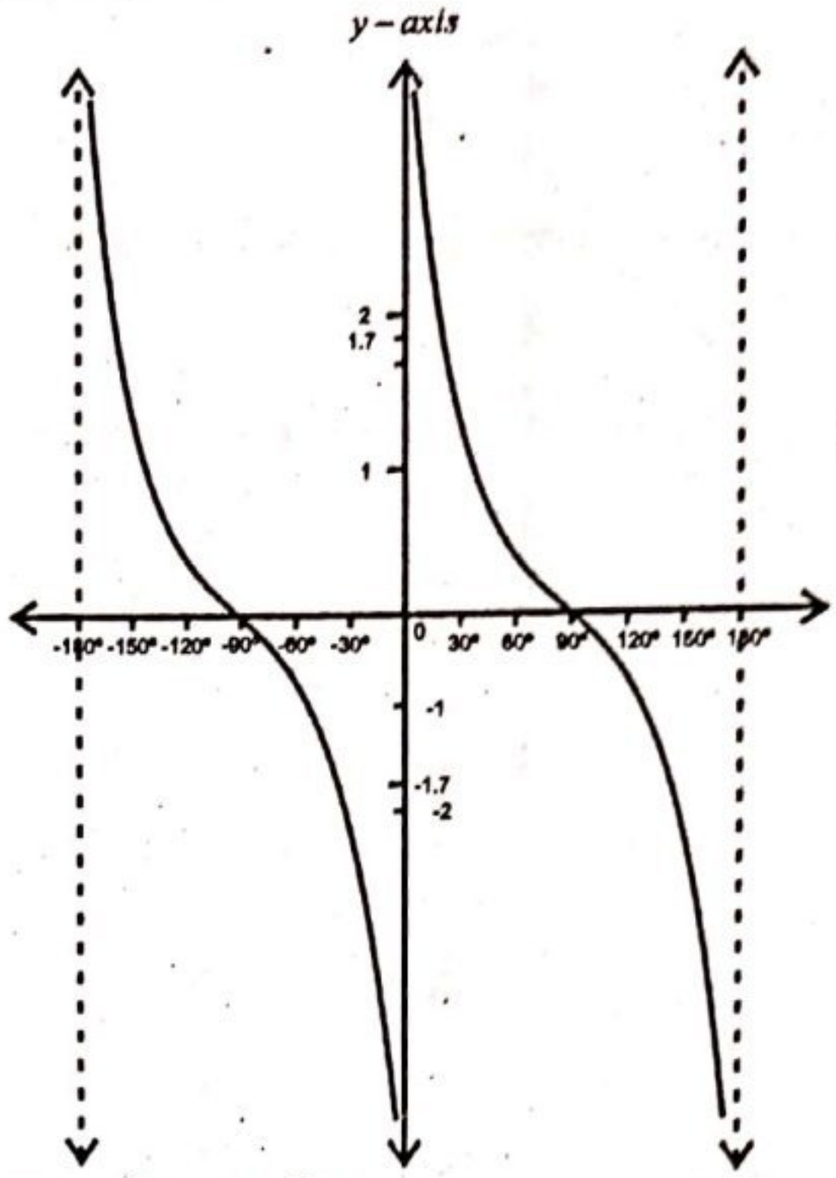

**3. Graph of $y = \tan x$ from $-\pi$ to $\pi$**



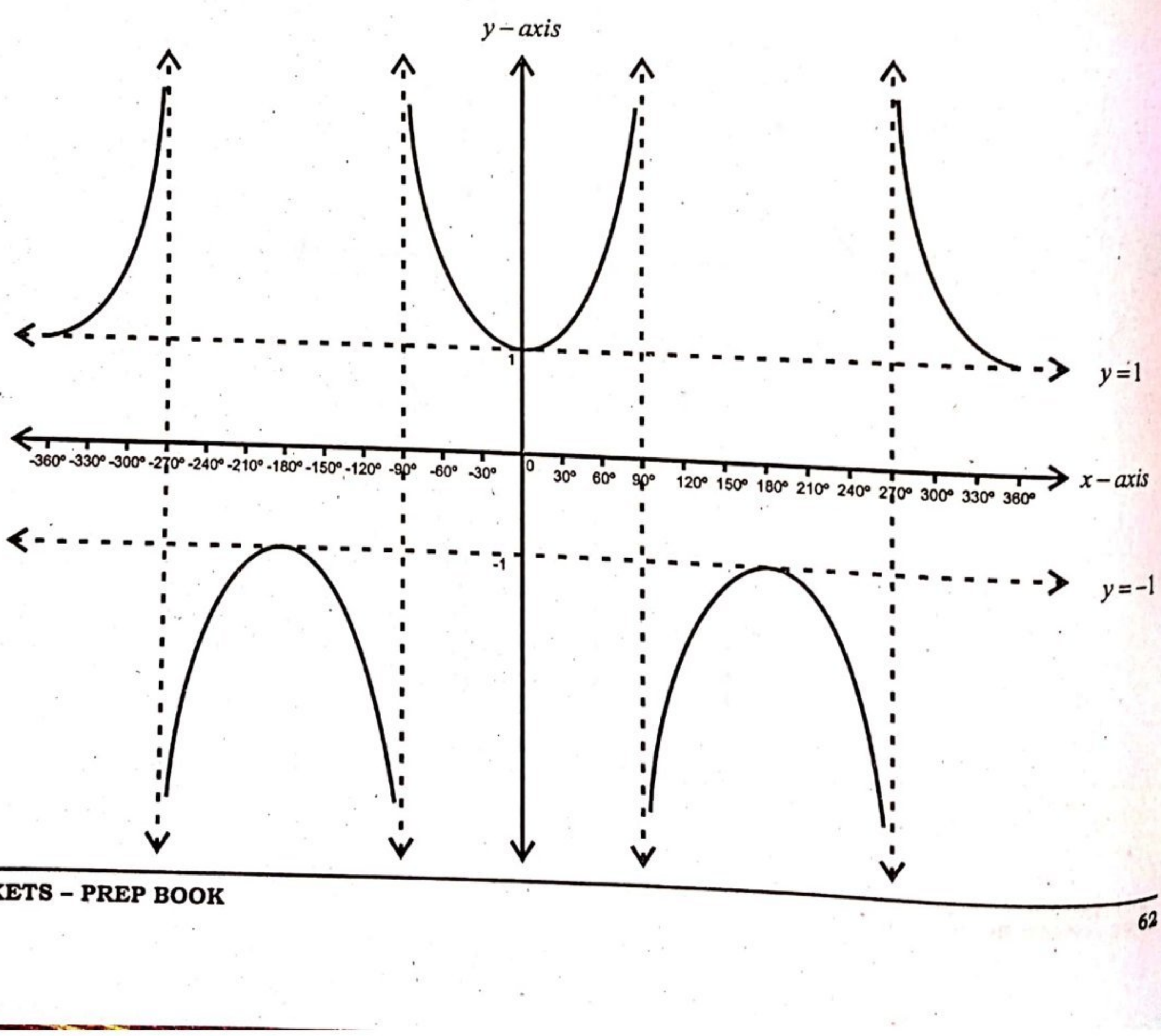



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**4. Graph of $y = \cot x$ from $-\pi$ to $\pi$**



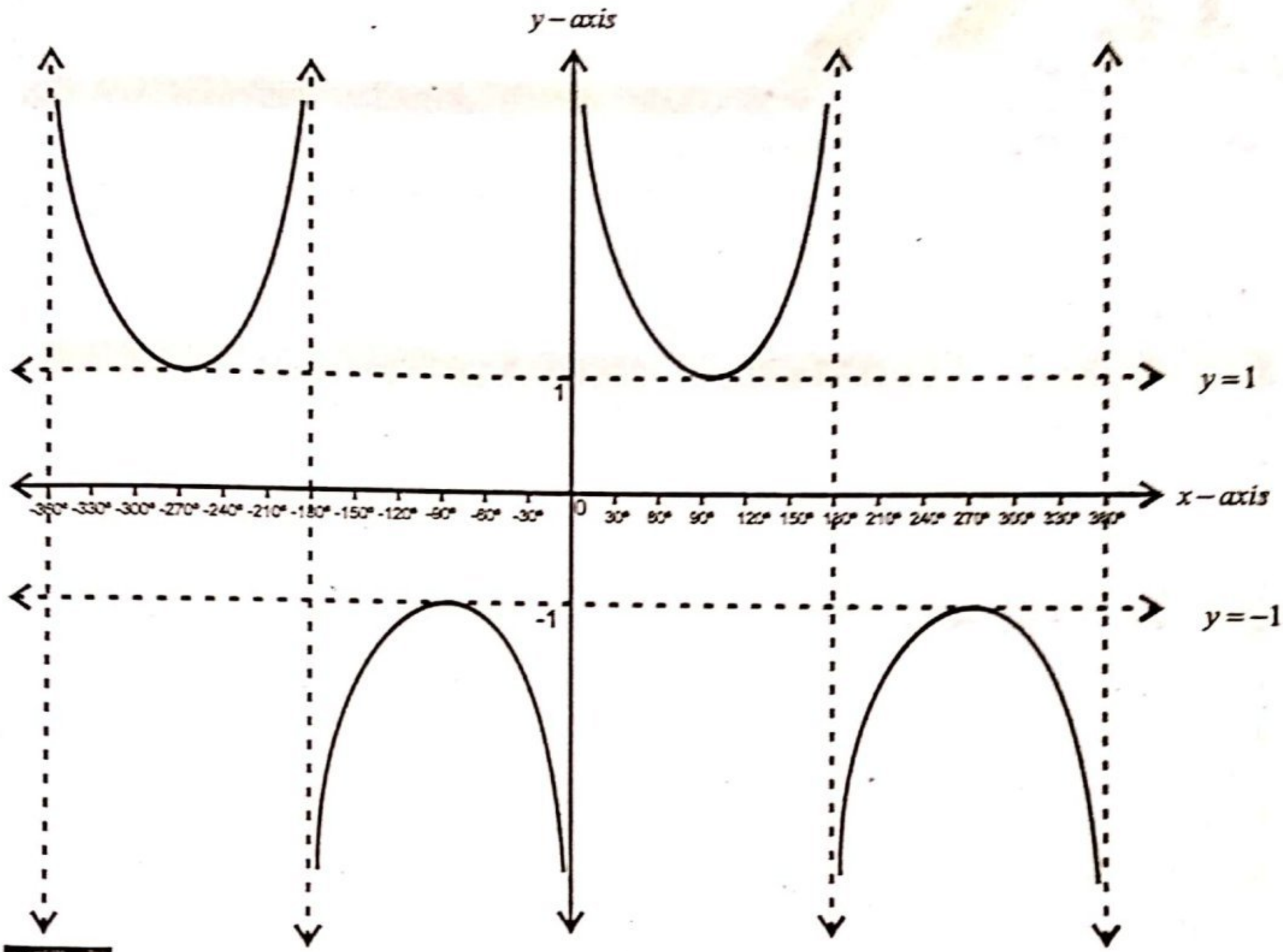

**5. Graph of $y = \sec x$ from $-2\pi$ to $2\pi$**






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6. Graph of $y = \operatorname{cosec} x$ from $-2\pi$ to $2\pi$




**Results:**

(i) Graph of trigonometric function are smooth curves.

(ii) The graphs of $\sin x$ and $\cos x$ are bounded graphs and bounded by the lines $y = \pm 1$ or $y \pm 1 = 0$

(iii) The graphs of $\tan x, \cot x, \sec x, \operatorname{cosec} x$ are unbounded.

(iv) The graph of $\sin x$, $\tan x$, $\operatorname{cosec} x$, $\cot x$ are symmetric about origin.

(v) The graph of $\cos x$ and $\sec x$ are symmetric about $y$-axis.

(vi) From graph we can find domain and range of trigonometric functions.

(vii) Vertical dashed lines in the graph of $\tan x$, $\cot x$, $\sec x$, $\operatorname{cosec} x$ are vertical asymptotes.

**Do you know?**
Graph of which trigonometric function does not meet $x$-axis, $y$-axis and origin?



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# 12 CHAPTER APPLICATION OF TRIGONOMETRY

## Elements of the Triangle *ABC*

- For a given triangle *ABC*, we shall use *a, b, c* to denote the measure of sides and A, B, C to denote the measures of angles.
- The quantities *a, b, c,* A, B and C are called the elements of the triangle *ABC*.

## Types of Triangles

| NAME OF TRIANGLE | SIDES | ANGLES |
|---|---|---|
| Scalene | 3 sides are different | 3 angles are different |
| Isosceles | 2 sides are same | 2 angles are same |
| Equilateral | 3 sides are same | 3 angles are same (*i.e.* of 60°) |
| Acute | May be scalene, isosceles or equilateral | 3 angles are acute (less than 90°) |
| Obtuse | May be scalene or isosceles but not equilateral | 1 angle is obtuse (greater than 90° but less than 180°) |
| Right | May be scalene or isosceles but not equilateral | 1 angle is of 90° |
| Oblique | May be scalene, isosceles or equilateral | No angle is of 90° |

## Solution of Triangles

- If any three out of six elements (with at least one side) are given the remaining three elements can be determined. This process of finding the unknown elements is called the **solution of the triangle.**

## Solution of Right Triangle

Triangle cannot be solved if only three angles are given.

- In order to solve a right triangle there are two cases

**Case – I**

When measures of one side and acute angle is given.

**Case – II**

When measure of two sides are given.

- Any right triangle can be solved by using **trigonometric ratios** or **Pythagoras theorem.**

## Heights and Distances

One of the chief advantages of trigonometry lies in finding heights and distances of inaccessible objects:

In order to solve such problems, the following procedure is adopted:

(i) Construct a clear labelled diagram, showing the known measurements.

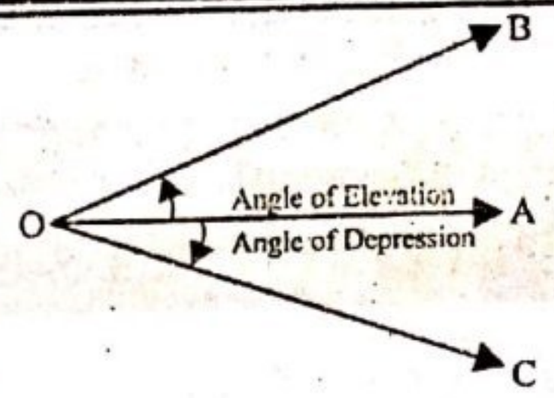
(ii) Establish the relationship between the quantities in the diagram to form equations containing trigonometric ratios.

(iii) Unless stated otherwise, the height of the observer is ignored.



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## Angles of Elevation and Depression




If $\overrightarrow{OA}$ is the horizontal ray through the eye of the observer at point O, and there are two objects $B$ and $C$ such that $B$ is above and $C$ is below the horizontal ray $\overrightarrow{OA}$, then

(i) For looking at $B$ above the horizontal ray, we have to raise our eye and $\angle AOB$ is called the **Angle of Elevation** and

(ii) For looking at $C$ below the horizontal ray we have to lower our eye and $\angle AOC$ is called the **Angle of Depression.**

> Angle of elevation and depression are always acute angles.

## Solution of Oblique Triangles

| ONE SIDE AND TWO ANGLES ARE GIVEN | TWO SIDES AND INCLUDED ANGLE IS GIVEN | THREE SIDES ARE GIVEN |
|---|---|---|
| Law of sines | (i) First law of cosine and then law of sines<br>(ii) First law of tangents and then law of sines | (i) The law of cosine<br>(ii) The half angle formulas. |

| THE LAW OF SINES | THE LAW OF COSINE | THE LAW OF TANGENTS |
|---|---|---|
| $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{a}{\sin C}$ | $a^2=b^2+c^2-2bc\cos A$<br>$b^2=a^2+c^2-2ac\cos B$<br>$c^2=a^2+b^2-2ab\cos C$ **or**<br>$\cos A=\frac{b^2+c^2-a^2}{2bc}$<br>$\cos B=\frac{a^2+c^2-b^2}{2ac}$<br>$\cos C=\frac{a^2+b^2-c^2}{2ab}$ | $\frac{a-b}{a+b}=\frac{\tan\left(\frac{A-B}{2}\right)}{\tan\left(\frac{A+B}{2}\right)}$, $\frac{b-c}{b+c}=\frac{\tan\left(\frac{B-C}{2}\right)}{\tan\left(\frac{B+C}{2}\right)}$,<br>$\frac{c-a}{c+a}=\frac{\tan\left(\frac{C-A}{2}\right)}{\tan\left(\frac{C+A}{2}\right)}$ |

> If $\Delta ABC$ is right triangle then law of cosine reduces to Pythagoras theorem.

## Half Angles Formulas

| SINE | COSINE | TANGENT |
|---|---|---|
| $\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$ | $\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$ | $\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$ |
| $\sin\frac{B}{2}=\sqrt{\frac{(s-a)(s-c)}{ac}}$ | $\cos\frac{B}{2}=\sqrt{\frac{s(s-b)}{ac}}$ | $\tan\frac{B}{2}=\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$ |
| $\sin\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{ab}}$ | $\cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}}$ | $\tan\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$ |



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## Area of Triangles

| TWO SIDES & THEIR INCLUDED ANGLE IS GIVEN | ONE SIDE AND TWO ANGLES ARE GIVEN | THREE SIDES ARE GIVEN |
|---|---|---|
| $\Delta = \frac{1}{2}bc\sin A$<br>$\Delta = \frac{1}{2}ac\sin B$<br>$\Delta = \frac{1}{2}ab\sin C$ | $\Delta = \frac{a^2 \sin B \sin C}{2\sin A}$<br>$\Delta = \frac{b^2 \sin A \sin C}{2\sin B}$<br>$\Delta = \frac{c^2 \sin A \sin B}{2\sin C}$ | $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$<br>This formula is called **Hero's formula.** |

- Usually the area of the triangle is denoted by $\Delta$
- In all above cases $s = \frac{a+b+c}{2}$, which is half of the perimeter of triangle.

## Circles Connected To Triangles

| CIRCUM CIRCLE | IN CIRCLE | EX-CIRCLE |
|---|---|---|
| • The circle passing through the vertices of a triangle is called a **circum circle.**<br>• The centre of circum circle is called **circum centre**. It is the point of intersection of the right bisectors of the sides of the triangle.<br>• The radius of circum circle is called **circum radius**<br>• It is denoted by "**R**"<br>$R = \frac{abc}{4\Delta}$<br>$R = \frac{a}{2\sin A}$<br>$R = \frac{b}{2\sin B}$<br>$R = \frac{c}{2\sin C}$ | • The circle drawn inside a triangle touching its three sides is called its **inscribed circle** or **in circle.**<br>• The centre of in-circle is called **in-centre**. It is the point of intersection of the bisectors of angles of the triangle.<br>• Its radius is called **in-radius**<br>• It is denoted by "**r**"<br>$r = \frac{\Delta}{s}$ | • A circle, which touches one side of the triangle externally and the other two produced sides, is called an **escribed circle** or **ex-circle** or **e-circle.**<br>• Three such circles can be drawn.<br>• The centers of these circles (are called **ex-centers**) are the points where the internal bisector of one and the external bisectors of the other two angles of the triangle meet.<br>• The radii of ex-circles opposite to the vertex A, B and C are denoted by $r_1$, $r_2$ and $r_3$ respectively.<br>$r_1 = \frac{\Delta}{s-a}$<br>$r_2 = \frac{\Delta}{s-b}$<br>$r_3 = \frac{\Delta}{s-c}$ |



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## Relations Between r, R, $r_1$, $r_2$, $r_3$, s and $\Delta$

(i) $r_1r_2 + r_2r_3 + r_3r_1 = s^2$

(ii) $rr_1r_2r_3 = \Delta^2$

(iii) $r_1 + r_2 + r_3 = 4R + r$

(iv) $r_1r_2r_3 = rs^2$

(v) $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$

Area of an equilateral triangle $= \frac{\sqrt{3}}{4}(\text{side})^2$

Following two relations are only for equilateral triangle.

(i) $r : R : r_1 = 1 : 2 : 3$

(ii) $r : R : r_1 : r_2 : r_3 = 1 : 2 : 3 : 3 : 3$

## Formation of Triangles

If the lengths of two sides of a triangle and the angle opposite to one of those sides are given, it is possible that two triangles, one triangle, or no triangle can be constructed with the data. This is called an ambiguous case. If the length of sides $a$ and $b$ and the value of $\angle A$ are given, the length of side $b$ determines the number of triangles that can be constructed.

**Case – I**

(i) ÿ $m\angle A > 90^\circ$ and $a \le b$ then no triangle can be formed.

(ii) If $m\angle A > 90^\circ$ and $a > b$ then only one obtuse triangle can be formed.

**Case – II**

(i) If $m\angle A < 90^\circ$ and $a < b.\sin A$ then no triangle can be formed.

(ii) If $m\angle A < 90^\circ$ and $a > b$ then only one triangle can be formed.

(iii) If $m\angle A < 90^\circ$ and $a = b.\sin A$ then only one triangle can be formed.

(iv) If $m\angle A < 90^\circ$ and $b.\sin A < a < b$ then two triangles can be formed.



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# 13 CHAPTER

# INVERSE TRIGONOMETRIC FUNCTIONS

## Introduction

- We have already learned that only a bijective (one-to-one and onto) function will have an inverse function.
- If a function is not bijective, it may be possible to restrict its domain to make it bijective.

**Note:**
In General, no trigonometric functions are bijective due to their periodic behavior.

## Principal Trigonometric Function

If we restrict the domain of a trigonometric function in a specific interval to make it a bijective function, than it is called Principal trigonometric function.
Now it becomes possible to find inverse of a trigonometric function.

## Domains and Ranges of Principal Trigonometric Functions

| TRIGONOMETRIC FUNCTION | DOMAIN | RANGE |
|---|---|---|
| $y=\sin x$ | $-\frac{\pi}{2}\le x\le\frac{\pi}{2}$ | $-1\le y\le 1$ |
| $y=\cos x$ | $0\le x\le\pi$ | $-1\le y\le 1$ |
| $y=\tan x$ | $-\frac{\pi}{2}<x<\frac{\pi}{2}$ | $y\in\mathbb{R}$ or $(-\infty,\infty)$ |
| $y=\cot x$ | $0<x<\pi$ | $y\in\mathbb{R}$ or $(-\infty,\infty)$ |
| $y=\sec x$ | $[0,\pi], x\neq\frac{\pi}{2}$ | $y\le -1$ or $y\ge 1$ |
| $y=\cos ecx$ | $\left[-\frac{\pi}{2},\frac{\pi}{2}\right], x\neq 0$ | $y\le -1$ or $y\ge 1$ |

## Domains and Ranges of Inverse Trigonometric Functions

| INVERSE TRIGONOMETRIC FUNCTION | DOMAIN | RANGE |
|---|---|---|
| $y=\sin^{-1}x$ | $-1\le x\le 1$ | $-\frac{\pi}{2}\le y\le\frac{\pi}{2}$ |
| $y=\cos^{-1}x$ | $-1\le x\le 1$ | $0\le y\le\pi$ |
| $y=\tan^{-1}x$ | $x\in\mathbb{R}$ or $(-\infty,\infty)$ | $-\frac{\pi}{2}<y<\frac{\pi}{2}$ |
| $y=\cot^{-1}x$ | $x\in\mathbb{R}$ or $(-\infty,\infty)$ | $0<y<\pi$ |
| $y=\sec^{-1}x$ | $x\le -1$ or $x\ge 1$ | $[0,\pi], y\neq\frac{\pi}{2}$ |
| $y=\cos ec^{-1}x$ | $x\le -1$ or $x\ge 1$ | $\left[-\frac{\pi}{2},\frac{\pi}{2}\right], y\neq 0$ |



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**Note:**

- Principal trigonometric functions have no period.
- The graph of a function and its inverse always reflect about the line $y \equiv x$.
- Inverse trigonometric function is also read as "arc function". Why?

**Do You know?**

(i) $\sin^{-1}x \neq (\sin x)^{-1}$

(ii) $\cos^{-1}x \neq (\cos x)^{-1}$

(iii) $\tan^{-1}x \neq (\tan x)^{-1}$

(iv) $sec^{-1}x \neq (secx)^{-1}$

(v) $\cos ec^{-1}x \neq (\cos ecx)^{-1}$

(vi) $\cot^{-1}x \neq (\cot x)^{-1}$

## Properties of Inverse Trigonometric Functions

(i) $\cos ec^{-1}x = \sin^{-1}\left(\frac{1}{x}\right)$, $x \leq -1$ or $x \geq 1$

(ii) $\sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right)$, $x \leq -1$ or $x \geq 1$

(iii) $\cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right)$, $x > 0$

(iv) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$, $-1 \leq x \leq 1$

(v) $\cos ec^{-1}x + \sec^{-1}x = \frac{\pi}{2}$, $x \leq -1$ or $x \geq 1$

(vi) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$, $x \in \mathbb{R}$

(vii) $2\sin^{-1}x = \cos^{-1}(1-2x^2)$

(viii) $2\cos^{-1}x = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$

(ix) $2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

**Some Important Results:**

(i) $\sin^{-1}(-x) = -\sin^{-1}x$, $-1 \leq x \leq 1$

(ii) $\cos^{-1}(-x) = \pi - \cos^{-1}x$, $-1 \leq x \leq 1$

(iii) $\tan^{-1}(-x) = -\tan^{-1}x$, $x \in \mathbb{R}$

(iv) $\cos ec^{-1}(-x) = -\cos ec^{-1}x$, $x \leq -1$ or $x \geq 1$

(v) $\sec^{-1}(-x) = \pi - \sec^{-1}x$, $x \leq -1$ or $x \geq 1$

(vi) $\cot^{-1}(-x) = \pi - \cot^{-1}x$, $x \in \mathbb{R}$

(vii) $\sin^{-1}(\sin x) = x$ ; $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

(viii) $\cos^{-1}(\cos x) = x$ ; $0 \leq x \leq \pi$

(ix) $\tan^{-1}(\tan x) = x$ ; $-\frac{\pi}{2} < x < \frac{\pi}{2}$

(x) $\cot^{-1}(\cot x) = x$ ; $0 < x < \pi$

(xi) $\sec^{-1}(\sec x) = x$ ; $0 \leq x \leq \pi$, $x \neq \frac{\pi}{2}$



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**(xii)** $\operatorname{cosec}^{-1}(\operatorname{cosec} x) = x \; ; -\frac{\pi}{2} \le x \le \frac{\pi}{2}, \; x \neq 0$

**(xiii)** $\sin(\sin^{-1} x) = x \; ; -1 \le x \le 1$

**(xiv)** $\cos(\cos^{-1} x) = x \; ; -1 \le x \le 1$

**(xv)** $\tan(\tan^{-1} x) = x \; ; x \in \mathbb{R}$

**(xvi)** $\cot(\cot^{-1} x) = x \; ; x \in \mathbb{R}$

**(xvii)** $\sec(\sec^{-1} x) = x \; ; x \le -1 \text{ or } x \ge 1$

**(xviii)** $\operatorname{cosec}(\operatorname{cosec}^{-1} x) = x \; ; x \le -1 \text{ or } x \ge 1$

## Inverse Trigonometric Functions in Terms of Other Inverse Trigonometric Functions

| | | | | | |
|---|---|---|---|---|---|
| $\sin^{-1} x =$ | $\cos^{-1}\sqrt{1-x^2}$ | $\tan^{-1}\frac{x}{\sqrt{1-x^2}}$ | $\cot^{-1}\frac{\sqrt{1-x^2}}{x}$ | $\sec^{-1}\frac{1}{\sqrt{1-x^2}}$ | $\operatorname{cosec}^{-1}\frac{1}{x}$ |
| $\cos^{-1} x =$ | $\sin^{-1}\sqrt{1-x^2}$ | $\tan^{-1}\frac{\sqrt{1-x^2}}{x}$ | $\cot^{-1}\frac{x}{\sqrt{1-x^2}}$ | $\sec^{-1}\frac{1}{x}$ | $\operatorname{cosec}^{-1}\frac{1}{\sqrt{1-x^2}}$ |
| $\tan^{-1} x =$ | $\sin^{-1}\frac{x}{\sqrt{1+x^2}}$ | $\cos^{-1}\frac{1}{\sqrt{1+x^2}}$ | $\cot^{-1}\frac{1}{x}$ | $\sec^{-1}\sqrt{1+x^2}$ | $\operatorname{cosec}^{-1}\frac{\sqrt{1+x^2}}{x}$ |
| $\cot^{-1} x =$ | $\sin^{-1}\frac{1}{\sqrt{1+x^2}}$ | $\cos^{-1}\frac{x}{\sqrt{1+x^2}}$ | $\tan^{-1}\frac{1}{x}$ | $\sec^{-1}\frac{\sqrt{1+x^2}}{x}$ | $\operatorname{cosec}^{-1}\sqrt{1+x^2}$ |
| $\sec^{-1} x =$ | $\sin^{-1}\frac{\sqrt{x^2-1}}{x}$ | $\cos^{-1}\frac{1}{x}$ | $\tan^{-1}\sqrt{x^2-1}$ | $\cot^{-1}\frac{1}{\sqrt{x^2-1}}$ | $\operatorname{cosec}^{-1}\frac{x}{\sqrt{x^2-1}}$ |
| $\operatorname{cosec}^{-1} x =$ | $\sin^{-1}\frac{1}{x}$ | $\cos^{-1}\frac{\sqrt{x^2-1}}{x}$ | $\tan^{-1}\frac{1}{\sqrt{x^2-1}}$ | $\cot^{-1}\sqrt{x^2-1}$ | $\sec^{-1}\frac{x}{\sqrt{x^2-1}}$ |

## Addition and Subtraction Formulae

**(i)** $\sin^{-1} A + \sin^{-1} B = \sin^{-1}\left(A\sqrt{1-B^2} + B\sqrt{1-A^2}\right)$

**(ii)** $\sin^{-1} A - \sin^{-1} B = \sin^{-1}\left(A\sqrt{1-B^2} - B\sqrt{1-A^2}\right)$

**(iii)** $\cos^{-1} A + \cos^{-1} B = \cos^{-1}\left(AB - \sqrt{(1-A^2)(1-B^2)}\right)$

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**(iv)** $\cos^{-1}A-\cos^{-1}B=\cos^{-1}\left(AB+\sqrt{\left(1-A^2\right)\left(1-B^2\right)}\right)$

**(v)** $\tan^{-1}A+\tan^{-1}B=\begin{cases}\tan^{-1}\left(\dfrac{A+B}{1-AB}\right),\left(\text{where } A\geq 0,B\geq 0;AB<1\right)\\ \pi+\tan^{-1}\left(\dfrac{A+B}{1-AB}\right),\left(\text{where } A\geq 0,B\geq 0;AB>1\right)\end{cases}$

**(vi)** $\tan^{-1}A-\tan^{-1}B=\tan^{-1}\left(\dfrac{A-B}{1+AB}\right)$

**Do you know?**

- $\sin^{-1}A+\sin^{-1}B=\dfrac{\pi}{2}$ if $A^2+B^2=1$
- $\cos^{-1}A+\cos^{-1}B=\dfrac{\pi}{2}$ if $A^2+B^2=1$
- $\tan^{-1}A+\tan^{-1}B=\dfrac{\pi}{2}$ if $AB=1$ and $A,B>0$

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# 14 CHAPTER SOLUTIONS OF TRIGONOMETRIC EQUATIONS

## Trigonometric Equations

An equation involving at least one trigonometric ratio of unknown angle is called trigonometric equation. e.g, $2\sin x = 1$ ; $\cos^2 x = \frac{1}{4}$

## Reference Angle

The reference angle is the smallest positive angle between the terminal side and $x$-axis.

## Solution of a Trigonometric Equation

The values of the variables which satisfy the given trigonometric equation are called the solution of a trigonometric equation.

- Solution of a trigonometric equation is of two types:
  - (i) **Principal solution**
  - (ii) **General solution**

## Principal Solution

Every trigonometric equation has a principal solution which is defined as the principal value of the inverse trigonometric function.

## General Solution

A solution of a trigonometric equation generalized by means of periodicity is known as the general solution.

| TRIGONOMETRIC FUNCTIONS | PRINCIPAL VALUE | GENERAL VALUES |
|---|---|---|
| $\sin\theta$ | $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ | $\{\theta + 2n\pi\} \cup \{(\pi - \theta) + 2n\pi\}$ |
| $\cos\theta$ | $\theta \in [0, \pi]$ | $\{\theta + 2n\pi\} \cup \{(2\pi - \theta) + 2n\pi\}$ |
| $\tan\theta$ | $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ | $\{\theta + n\pi\} \cup \{(\pi + \theta) + n\pi\}$ |

**Some Important Results:**

| TRIGONOMETRIC EQUATION | PRINCIPAL SOLUTION | GENERAL SOLUTIONS |
|---|---|---|
| $\sin x = 0$ | $x = 0$ | $x = n\pi, n \in \mathbb{Z}$ |
| $\cos x = 0$ | $x = \frac{\pi}{2}$ | $x = (2n+1)\frac{\pi}{2}, n \in \mathbb{Z}$ |
| $\tan x = 0$ | $x = 0$ | $x = n\pi, n \in \mathbb{Z}$ |

## Important Points to Remember

- Avoid cancellation of the common factors from both sides of the equation.
- If square on both sides of the equation is taken, then check for non-extraneous roots.



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KIPS

ENTRY TESTS

SERIES

PREP BOOK

# MATHEMATICS

FUNG

- Chapter-wise Complete Syllabus
- Comprehensive Course Revision
- Detailed Explanation of Topics
- Key Points, Tables, Flow Sheet & Graphs
- Easy to Remember; Important Key Facts

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# BOOK-II

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# 1 CHAPTER FUNCTIONS AND LIMITS

## Introduction

| | |
|---|---|
| **Function /Mapping*** | (i) A function is a rule or correspondence, relating two sets in such a way that each element in the first set corresponds to one and only one element in the second set.<br>(ii) A function $f$ from a set $X$ to $Y$ is a rule or a correspondence that assigns to each element $x$ in $X$ a unique element $y$ in $Y$. |
| **Notation for a Function** | (i) $y = f(x)$<br>(ii) $f : X \to Y$<br>(iii) $f$ is a mapping from $X$ to $Y$ |
| **Domain** | From the above notation $X$ is called domain and variable $x$ is called independent variable of $f(x)$. |
| **Range** | The set of images is called range. and variable $y$ is called dependent variable of $f(x)$. |
| **Graph of a Function** | If a vertical line intersects a graph in at most one point, it is the graph of a function. |
| **Piece-Wise Functions** | Such functions which are defined by two or more rules are called piece-wise defined functions. |

## Types of Functions

| 1. ALGEBRAIC FUNCTIONS | DEFINITION | DOMAIN/RANGE |
|---|---|---|
| **Polynomial Function** | The function of the form $P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + ax + a_0$ is called polynomial function of degree $n$. $a_n \neq 0$ , and $n$ is any non negative integer. | Domain = $\mathbb{R}$<br>Range = $\mathbb{R}$ |
| **Linear Function** | If the degree of a polynomial function is 1, then it is called linear function.<br>The general form a linear function is $f(x) = ax + b \quad a \neq 0, a \,\&\, b$ are real numbers.<br>The graph of a linear function is a straight line | Domain = $\mathbb{R}$<br>Range = $\mathbb{R}$ |
| **Identity Function** | For any set $X$, a function $I : X \to X$ of the form $I(x) = x$, $\forall x \in X$ is called an identity function. | Domain = $X$ (Itself)<br>Range = $X$ (Itself) |

* The term function was recognized by a German Mathematician Leibniz (1646-1716).



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| | | |
|---|---|---|
| **Constant Function** | Let X and Y be sets of real numbers. A function $C: X \to Y$ defined by $C(x) = a, \forall x \in X; a \in Y$ is called a constant function. | Domain = $\mathbb{R}$<br>Range = $\{a\}$ |
| **Rational Function** | A function $R(x)$ of the form $\frac{P(x)}{Q(x)}$, where both $P(x)$ and $Q(x)$ are polynomial functions and $Q(x) \neq 0$, is called a rational function. | Domain = $\{x \mid x \in \mathbb{R} \wedge Q(x) \neq 0\}$<br>Range = Subset of real numbers. |

| 2. TRIGONOMETRIC FUNCTIONS | DOMAIN | RANGE |
|---|---|---|
| $y = \sin x$ | $\mathbb{R}$ | $-1 \leq y \leq 1$ |
| $y = \cos x$ | $\mathbb{R}$ | $-1 \leq y \leq 1$ |
| $y = \tan x$ | $\left\{x \mid x \in \mathbb{R}, x \neq \frac{(2n+1)\pi}{2}, \ n \in \mathbb{Z}\right\}$ | $\mathbb{R}$ |
| $y = \cot x$ | $\{x \mid x \in \mathbb{R}, x \neq n\pi, \ n \in \mathbb{Z}\}$ | $\mathbb{R}$ |
| $y = \sec x$ | $\left\{x \mid x \in \mathbb{R}, \ x \neq (2n+1)\frac{\pi}{2}, \ n \in \mathbb{Z}\right\}$ | $\{y \mid y \geq 1 \text{ or } y \leq -1\}$ |
| $y = \operatorname{cosec} x$ | $\{x \mid x \in \mathbb{R}, x \neq n\pi, \ n \in \mathbb{Z}\}$ | $\{y \mid y \geq 1 \text{ or } y \leq -1\}$ |

| 3. INVERSE TRIGONOMETRIC FUNCTIONS | DOMAIN | RANGE |
|---|---|---|
| $y = \sin^{-1} x$ | $-1 \leq x \leq 1$ | $\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$ |
| $y = \cos^{-1} x$ | $-1 \leq x \leq 1$ | $0 \leq y \leq \pi$ |
| $y = \tan^{-1} x$ | $(-\infty, \infty)$ or $\mathbb{R}$ | $\frac{-\pi}{2} < y < \frac{\pi}{2}$ |
| $y = \cot^{-1} x$ | $(-\infty, \infty)$ or $\mathbb{R}$ | $0 < y < \pi$ |
| $y = \sec^{-1} x$ | $x \leq -1$ *or* $x \geq 1$ | $[0, \pi], y \neq \frac{\pi}{2}$ |
| $y = \operatorname{cosec}^{-1} x$ | $x \leq -1$ *or* $x \geq 1$ | $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right], y \neq 0$ |

## 4. Exponential Function

A function of the form $f(x) = a^x, \ a > 0, a \neq 1$ defines an exponential function for each different constant $a$, called the base.

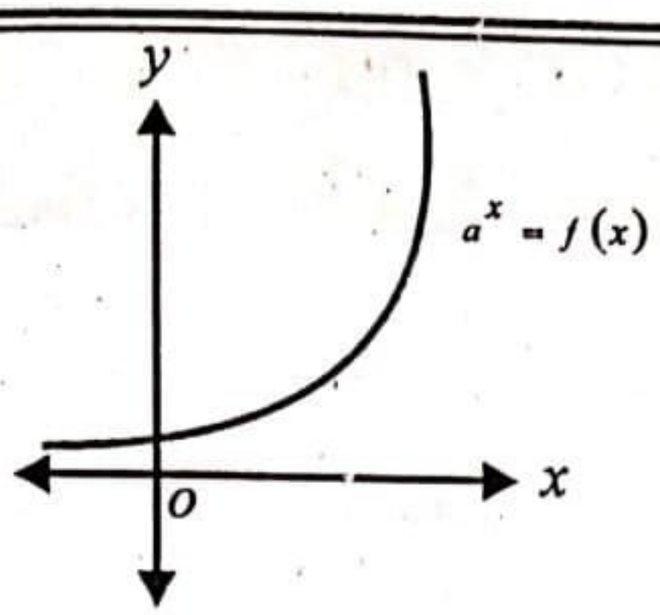
**Domain** = $\mathbb{R}$

**Range** = $(0, \infty)$

(i) Exponential function never assume the value zero.
(ii) The graph of exponential function always lies in I and II quadrant.



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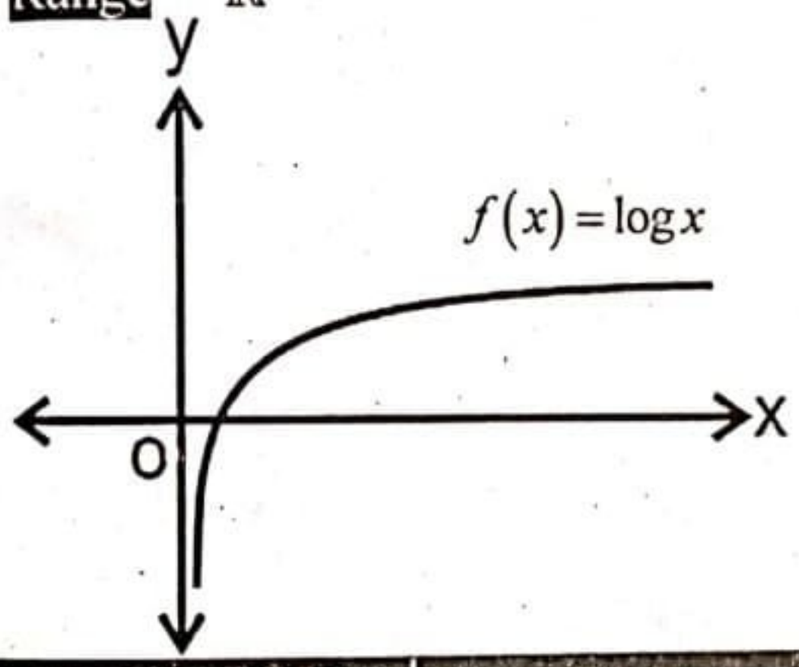

The exponential function and logarithmic function are inverses of each other.

## 5. Logarithmic Function

If $x = a^y$, then $y = \log_a x$, where $a > 0, a \neq 1$ is called logarithmic function of $x$.

**Domain** $= (0, \infty)$

**Range** $= \mathbb{R}$




(i) If a = 10, then we have $\log_{10} x$ which is called common logarithm of $x$.

(ii) If $a = e$ then we have $\log_e x$ which is called natural logarithm of $x$ denoted by $\ln x$.

(iii) The graph of logarithmic function lies in I and IV quadrant.

| 6. HYPERBOLIC FUNCTION | DEFINITION | DOMAIN | RANGE |
|---|---|---|---|
| $y = \sinh x$ | $\frac{1}{2}\left(e^x - e^{-x}\right)$ | $\mathbb{R}$ | $\mathbb{R}$ |
| $y = \cosh x$ | $\frac{1}{2}\left(e^x + e^{-x}\right)$ | $\mathbb{R}$ | $[1, \infty)$ |
| $y = \tanh x$ | $\frac{e^x - e^{-x}}{e^x + e^{-x}}$ | $\mathbb{R}$ | $(-1, 1)$ |
| $y = \coth x$ | $\frac{e^x + e^{-x}}{e^x - e^{-x}}$ | $\mathbb{R} - \{0\}$ | $\mathbb{R} - [-1,1]$ |
| $y = \sec hx$ | $\frac{2}{e^x + e^{-x}}$ | $\mathbb{R}$ | $(0,1]$ |
| $y = \text{cosech}\, x$ | $\frac{2}{e^x - e^{-x}}$ | $\mathbb{R} - \{0\}$ | $\mathbb{R} - \{0\}$ |

## Hyperbolic Identities

(i) $\cosh^2 x - \sinh^2 x = 1$

(ii) $\cosh^2 x + \sinh^2 x = \cosh 2x$

(iii) $\sinh 2x = 2 \sinh x \cosh x$

(iv) $\coth^2 x - 1 = \text{cosec}\, h^2 x$

(v) $1 - \tanh^2 x = \text{sech}^2 x$



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| 7. INVERSE HYPERBOLIC FUNCTION | DEFINITION | DOMAIN | RANGE |
|---|---|---|---|
| $y=\sinh^{-1}x$ | $\ell n\left(x+\sqrt{x^2+1}\right)$ | $\mathbb{R}$ | $\mathbb{R}$ |
| $y=\cosh^{-1}x$ | $\ell n\left(x+\sqrt{x^2-1}\right)$ | $[1,\infty)$ | $(0,\infty)$ |
| $y=\tanh^{-1}x$ | $\frac{1}{2}\ell n\left(\frac{1+x}{1-x}\right)$ | $(-1,1)$ | $\mathbb{R}$ |
| $y=\coth^{-1}x$ | $\frac{1}{2}\ell n\left(\frac{x+1}{x-1}\right)$ | $\mathbb{R}-[-1,1]$ | $\mathbb{R}-\{0\}$ |
| $y=\sec h^{-1}x$ | $\ell n\left(\frac{1}{x}+\frac{\sqrt{1-x^2}}{x}\right)$ | $(0,1]$ | $[0,\infty)$ |
| $y=\operatorname{cosech}^{-1}x$ | $\ell n\left(\frac{1}{x}+\frac{\sqrt{1+x^2}}{\lvert x\rvert}\right)$ | $\mathbb{R}-\{0\}$ | $\mathbb{R}-\{0\}$ |

## Some Other Important Functions

| NAME OF THE FUNCTION | DEFINITION | EXAMPLE |
|---|---|---|
| Explicit Function | If $y$ is easily expressed in terms of independent variable $x$, then $y$ is called an explicit function. Symbolically written as $y=f(x)$ | $y=\sqrt{2x+1}$<br>$y=x^2+4x+1$ |
| Implicit Function | If $y$ cannot expressed in terms of independent variable $x$, then $y$ is called an implicit function. Symbolically written as $f(x,y)=0$ | $xy^2+2x^2y=0$ |
| Even Function | A function $f(x)$ is said to be even if $f(-x)=f(x)$.<br>The graph of an even function is symmetric about $y-axis$. | $f(x)=\cos x$<br>$f(x)=x^2+1$ |
| Odd Function | A function $f(x)$ is said to be odd if $f(-x)=-f(x)$.<br>The graph of an odd function is symmetric about origin. | $f(x)=\sin x$<br>$f(x)=x^3+x$ |
| Absolute Value Function | $\lvert x\rvert=\begin{cases}x & x\geq 0\\ -x, & x<0\end{cases}$ | $\lvert 3\rvert=3$<br>$\lvert -4.12\rvert=4.12$ |



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| | | |
|---|---|---|
| **Parametric Functions** | Sometimes the relation between $x$ and $y$ can be expressed in terms of a third variable, which is called parameter, such types of functions are parametric functions. | $\left.\begin{aligned} x &= at^2 \\ y &= at \end{aligned}\right\}$ Parabola<br>$\left.\begin{aligned} x &= a\cos\theta \\ y &= a\sin\theta \end{aligned}\right\}$ Circle<br>$\left.\begin{aligned} x &= a\cos\theta \\ y &= b\sin\theta \end{aligned}\right\}$ Ellipse<br>$\left.\begin{aligned} x &= a\sec\theta \\ y &= b\tan\theta \end{aligned}\right\}$ Hyperbola |

## Algebraic Operations on Functions

| NAME OF OPERATION | DEFINITION |
|---|---|
| **Composition of Functions** | Let $f$ be a function from set $X$ to set $Y$ and $g$ be a function from set $Y$ to set $Z$. The composition of $f$ and $g$ is a function, denoted by $gof$, from $X$ to $Z$ and is defined by<br>$(gof)(x) = g(f(x)) = gf(x), \quad \forall x \in X$<br>Remember that,<br>**(i)** $fog(x) \neq gof(x)$ in general.<br>**(ii)** $f^2(x) = f(f(x))$<br>**(iii)** $f^3(x) = f(f(f(x)))$ and so on. |
| **Inverse of a Function** | Let $f$ be a one-one function from $X$ onto $Y$. The inverse function of $f$, denoted by $f^{-1}$, is a function from $Y$ onto $X$ and is defined by:<br>$x = f^{-1}(y), \forall y \in Y$ if and only if $y = f(x), \ \forall\ x \in X$.<br>**(i)** $fof^{-1}(x) = f^{-1}of(x) = x$<br>**(ii)** Remember that inverse of a function may or may not be a function. |

## Limits

| | |
|---|---|
| **Limit of a Function** | A function $f(x)$ has the limit "$L$" as $x \to a$ ($x$ approaches to $a$) if for every positive number $\varepsilon$, there is a positive number $\delta$ such that $|f(x) - L| < \varepsilon$ if $0 < |x - a| < \delta$.<br>Symbolically $\lim\limits_{x \to a} f(x) = L$<br>The $\lim\limits_{x \to a} f(x)$ may exist even if $f(x)$ is not defined at $x = a$. |



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## Theorems on Limits of Functions

(i) $\lim_{x\to a}\left[f(x)\pm g(x)\right]=\lim_{x\to a}f(x)\pm\lim_{x\to a}g(x)$

(ii) $\lim_{x\to a}\left[f(x).g(x)\right]=\lim_{x\to a}f(x).\lim_{x\to a}g(x)$

(iii) $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a}f(x)}{\lim_{x\to a}g(x)}$, $\lim_{x\to a}g(x)\neq 0$

(iv) $\lim_{x\to a}\left(Kf(x)\right)=K\lim_{x\to a}f(x)$, $K$ is any real number.

(v) If $f(x)\le g(x)$ for all $x$, then $\lim_{x\to a}f(x)\le\lim_{x\to a}g(x)$.

(vi) $\lim_{x\to a}\ln f(x)=\ln\lim_{x\to a}f(x)$

(vii) $\lim_{x\to a}\left[f(x)\right]^n=\left[\lim_{x\to a}f(x)\right]^n$, where $n$ is an integer.

## Sandwitch Theorem (Squeeze Principle)

Let $f$, $g$ and $h$ be functions such that $f(x)\le g(x)\le h(x)$ for all values of $x$. if $\lim_{x\to a}f(x)=L$ and $\lim_{x\to a}h(x)=L$, then $\lim_{x\to a}g(x)=L$.

## Limit of a Function at Infinity

(a) The limit of a function at infinity means the limit as $x\to+\infty$ ($x$ increases beyond all bounds). If the limit is $K$ we write as follows,
$\lim_{x\to+\infty}f(x)=K$.

(b) If $x\to-\infty$ ($x$ decreases beyond all bounds) and the limit of the function is L we write as follows, $\lim_{x\to-\infty}f(x)=L$.

## Indeterminate Forms

An indeterminate form is an expression of the type:

| | | |
|---|---|---|
| $\infty-\infty$ | $\frac{0}{0}$ | $\frac{\infty}{\infty}$ |
| $(\infty)^0$ | $(0)^\infty$ | $(0)^0$ |
| $(1)^\infty$ | $0\times\infty$ | $\infty\times\infty$ |

## Limits Involving Trigonometric Functions

- $\lim_{x\to 0}\frac{\sin x}{x}=1$
- $\lim_{x\to 0}\frac{\sin px}{qx}=\frac{p}{q}$, $p$ & $q$ are any real numbers.
- $\lim_{x\to 0}\frac{\tan x}{x}=1$
- $\lim_{x\to\infty}\frac{\sin x}{x}=0$
- $\lim_{x\to 0}\cos x=1$
- $\lim_{x\to 0}\sin x=0$

**Note:**

In all above cases $x$ is always measured in radians.



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## Important Formulae

- $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=e$
- $\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$ ·
- $\lim_{x\to\infty}\left(1+\frac{p}{x}\right)^{x}=e^{p}, p\in\mathbb{R}$
- $\lim_{x\to 0}\left(1+\frac{x}{a}\right)^{\frac{a}{x}}=e$
- $\lim_{x\to 0}(1+px)^{\frac{1}{x}}=e^{p}, p\in\mathbb{R}$
- $\lim_{x\to a}\frac{x^{n}-a^{n}}{x-a}=na^{n-1}$ $n$ is any integer.
- $\lim_{x\to\infty}e^{x}=\infty$
- $\lim_{x\to-\infty}e^{x}=0$
- $\lim_{x\to 0}e^{x}=1$
- $\lim_{x\to 0}\frac{a^{x}-1}{x}=\ln a$
- $\lim_{x\to 0}\frac{e^{x}-1}{x}=1$
- $\lim_{x\to 0}\frac{\ln(1+x)}{x}=1$

| ONE SIDED LIMITS | |
|---|---|
| The Left Hand Limit | $\lim_{x\to a^{-}}f(x)=L$ is read as the limit of $f(x)$ is equal to $L$ as $x$ approaches $a$ from the left i.e. for all $x$ sufficiently close to $a$, but less than $a$. |
| The Right Hand Limit | $\lim_{x\to a^{+}}f(x)=M$ is read as the limit of $f(x)$ is equal to $M$ as $x$ approaches $a$ from the right *i.e.* for all $x$ sufficiently close to $a$, but greater than $a$. |

## Existence of Limit of a Function

The limit of a function exists if left hand limit = right hand limit
i.e. if $\lim_{x\to a^{-}}f(x)=\lim_{x\to a^{+}}f(x)=L$

## Continuity of a Function at a Number

A function $f$ is said to be continuous at a number "c" *iff* the following three conditions are satisfied.

(i) $f(c)$ is defined.

(ii) $\lim_{x\to c}f(x)$ exists.



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(iii) $\lim_{x \to c} f(x) = f(c)$

If one or more of these conditions fail to hold at "c", then the function is **discontinuous.**

## Methods for Evaluating the Limit of a Function

| | |
|---|---|
| **Rationalization Method** | In this method, we rationalize the factor containing the square root. After simplification ,we put the value of $x$ to get the limit of $f(x)$ |
| **Factorization Method** | Let $\frac{f(x)}{g(x)}$, $g(x) \neq 0$, we factorize $f(x)$ and $g(x)$ if possible. Cancel the common factors to get the simplified form of $f(x)$. Now we put the value of $x$ to get the limit of $f(x)$. |
| **Substitution Method** | In this method we try out different forms of substitutions, depending upon the form of the given function and removing, in turn the indeterminate form. After simplification, we put the value of $x$ to get the limit of $f(x)$. |
| **Rule of De l'Hospital*** | *If* $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$ |

## To Check Symmetry

(i) **Symmetry about $y$-axis**

If we replace $(x, y)$ by $(-x, y)$ in the equation and no change occurs, then it is symmetric about $y-axis$.

(ii) **Symmetry about $x$-axis**

If we replace $(x, y)$ by $(x, -y)$ in the equation and no change occurs, then it is symmetric about $x-axis$.

(iii) **Symmetry about Origin**

If we replace $(x, y)$ by $(-x, -y)$ in the equation and no change occurs, then it is symmetric about origin.

* A French mathematician de l' Hospital (1661-1704)



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# 2 CHAPTER DIFFERENTIATION

| NAME OF MATHEMATICIAN | NOTATION USED FOR DERIVATIVE |
|---|---|
| Leibniz | $\frac{dy}{dx}$ or $\frac{df}{dx}$ |
| Newton | $\dot{f}(x)$ |
| Lagrange | $f'(x)$ |
| Cauchy | $Df(x)$ |

**Definition:**

For $y = f(x)$, the derivative of "$y$" with respect to $x$ is defined as instantaneous rate of change of '$y$' w.r.t. '$x$'. Symbolically

(i) $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

(ii) $\frac{d}{dx} f(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$

(iii) $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

$\frac{f(x+\delta x) - f(x)}{\delta x}$ is called average rate of change of $f(x)$ over the interval $[x, x+\delta x]$

The process of finding derivative is called differentiation.

## Theorems on Differentiation

(i) $\frac{d}{dx}(c) = 0$, where c is a constant

(ii) $\frac{d}{dx}(x) = 1$

(iii) $\frac{d}{dx}(cx) = c$

(iv) $\frac{d}{dx}(x^n) = nx^{n-1}$, $n \in \mathbb{R}$

(v) $\frac{d}{dx}(cx^n) = cnx^{n-1}$

(vi) $\frac{d}{dx}(u \pm v \pm w \pm \cdots) = \frac{du}{dx} \pm \frac{dv}{dx} \pm \frac{dw}{dx} \pm \cdots$ where $u = f(x), v = f(x), w = f(x)$

(vii) $\frac{d}{dx}(f(x).g(x)) = f'(x).g(x) + f(x).g'(x)$ (the product rule).

(viii) $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x).g'(x)}{(g(x))^2}$ (the quotient rule)

(ix) $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ (the chain rule)



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(x) $\frac{d}{dx}\left(\frac{1}{f(x)}\right)=-\frac{f'(x)}{[f(x)]^2}$ (reciprocal rule)

(xi) $\frac{d}{dx}(f(u))=f'(u)\frac{du}{dx}$

## Derivatives of Trigonometric Functions

(i) $\frac{d}{dx}\sin x=\cos x$

(ii) $\frac{d}{dx}\cos x=-\sin x$

(iii) $\frac{d}{dx}\tan x=\sec^2 x$

(iv) $\frac{d}{dx}\cot x=-\operatorname{cosec}^2 x$

(v) $\frac{d}{dx}\operatorname{cosec} x=-\operatorname{cosec} x\cot x$

(vi) $\frac{d}{dx}\sec x=\sec x\tan x$

## Derivatives of Inverse Trigonometric Functions

(i) $\frac{d}{dx}\sin^{-1}x=\frac{1}{\sqrt{1-x^2}},x\in(-1,1)$

(ii) $\frac{d}{dx}\cos^{-1}x=\frac{-1}{\sqrt{1-x^2}},x\in(-1,1)$

(iii) $\frac{d}{dx}\tan^{-1}x=\frac{1}{1+x^2},x\in\mathbb{R}$

(iv) $\frac{d}{dx}\cot^{-1}x=\frac{-1}{1+x^2},x\in\mathbb{R}$

(v) $\frac{d}{dx}\operatorname{cosec}^{-1}x=\frac{-1}{x\sqrt{x^2-1}},x\in[-1,1]'$

(vi) $\frac{d}{dx}\sec^{-1}x=\frac{1}{x\sqrt{x^2-1}},x\in[-1,1]'$

## Derivatives of Hyperbolic Functions

(i) $\frac{d}{dx}\sinh x=\cosh x$

(ii) $\frac{d}{dx}\cosh x=\sinh x$

(iii) $\frac{d}{dx}\tanh x=\operatorname{sech}^2 x$

(iv) $\frac{d}{dx}\coth x=-\operatorname{cosech}^2 x$

(v) $\frac{d}{dx}\operatorname{cosech} x=-\coth x\operatorname{cosech} x$



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(vi) $\frac{d}{dx}\operatorname{sech} x = -\tanh x \operatorname{sech} x$

## Derivatives of Inverse Hyperbolic Functions

(i) $\frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{1+x^2}}, x \in \mathbb{R}$

(ii) $\frac{d}{dx}\cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}, x > 1$

(iii) $\frac{d}{dx}\tanh^{-1} x = \frac{1}{1-x^2}, |x| < 1$

(iv) $\frac{d}{dx}\coth^{-1} x = \frac{1}{1-x^2}, |x| > 1$

(v) $\frac{d}{dx}\operatorname{sech}^{-1} x = \frac{-1}{x\sqrt{1-x^2}}, 0 < x < 1$

(vi) $\frac{d}{dx}\operatorname{cosech}^{-1} x = \frac{-1}{|x|\sqrt{1+x^2}}, x \in \mathbb{R} - \{0\}$

## Derivatives of Exponential and Logarithmic Functions

(i) $\frac{d}{dx}e^{f(x)} = e^{f(x)}.f'(x)$

(ii) $\frac{d}{dx}e^{x} = e^{x}$

(iii) $\frac{d}{dx}a^{f(x)} = a^{f(x)} \ln a.f'(x)$

(iv) $\frac{d}{dx}a^{x} = a^{x} \ln a$

(v) $\frac{d}{dx}\ln f(x) = \frac{1}{f(x)}.f'(x)$

(vi) $\frac{d}{dx}\ln x = \frac{1}{x}$

(vii) $\frac{d}{dx}\log_a f(x) = \frac{1}{f(x)}.\frac{1}{\ln a}f'(x)$

(viii) $\frac{d}{dx}\log_a x = \frac{1}{x \ln a}$



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## Higher Derivatives

The second, third and higher derivatives are defined as follows;

(i) Second derivative $=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2}=f''(x)=y''=D^2y=y_2$

(ii) Third derivative $=\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)=\frac{d^3y}{dx^3}=f'''(x)=y'''=D^3y=y_3$

(iii) $n^{th}$ derivative $=\frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right)=\frac{d^ny}{dx^n}=f^{(n)}(x)=y^{(n)}=D^ny=y_n$

## Series Expansions of Functions

**1. If $f$ is defined in the interval containing '$a$' and its derivatives of all orders exists at $x=a$, then, Taylor series expansion of a function $f(x)$ at $x=a$ is**

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2f''(a)}{2!}+\frac{(x-a)^3f'''(a)}{3!}+\frac{(x-a)^4f^{(4)}(a)}{4!}+\cdots$$

$$+\frac{(x-a)^nf^{(n)}(a)}{n!}+\cdots$$

Or

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)(x-a)^n}{n!}$$

**2. If $f$ is defined in the interval containing 0 and its derivatives exists at x = 0 then, Maclaurin Series expansion of a function is**

$$f(x)=f(0)+xf'(0)+x^2\frac{f''(0)}{2!}+x^3\frac{f'''(0)}{3!}+x^4\frac{f^{(4)}(0)}{4!}+\cdots+\frac{f^{(n)}(0)}{n!}x^n+\cdots$$

Or

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$$

## Some Important Maclaurin Series Expansions

| FUNCTION | MACLAURIN SERIES |
|---|---|
| $\sin x$ | $x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$ |
| $\cos x$ | $1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots$ |
| $\tan x$ | $x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}+\cdots$ |
| $e^x$ | $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots$ |
| $a^x$ | $1+x\ln a+\frac{x^2(\ln a)^2}{2!}+\frac{x^3(\ln a)^3}{3!}+\cdots$ |



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## Applications of Derivatives

| | |
|---|---|
| Geometrical Meaning of $\frac{dy}{dx}$ | If $y=f(x)$ then $\frac{dy}{dx}$ denotes the slope of the tangent to the given curve at any point $(x,y)$. |
| Equation of Tangent to $y=f(x)$ at $(x_1,y_1)$ | $y-y_1=\left(\frac{dy}{dx}\right)_{(x_1,y_1)}(x-x_1)$ |
| Equation of Normal to $y=f(x)$ at $(x_1,y_1)$ | $y-y_1=\frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1,y_1)}}(x-x_1)$ |
| Increasing Function | **(a)** Let a function $f(x)$ is defined in the interval $(a,b)$,then $f(x)$ is said to be increasing if $\forall x_1,x_2\in(a,b), x_1>x_2\Rightarrow f(x_1)>f(x_2)$<br>**(b)** If $f'(x)>0\forall x\in(a,b)$ |
| Decreasing Function | **(c)** Let a function $f(x)$ is defined in the interval $(a,b)$,then $f(x)$ is said to be decreasing if $\forall x_1,x_2\in(a,b), x_1>x_2\Rightarrow f(x_1)<f(x_2)$<br>**(d)** $f'(x)<0\forall x\in(a,b)$ |
| Extreme Values Or Maximum Values and Minimum Values Or Maxima and Minima | Let f be a differentiable function in a neighborhood of c where $f'(c)=0$ ,then<br>**1.** $f$ has relative maxima at c if $f''(c)<0$.<br>**2.** $f$ has relative minima at c if $f''(c)>0$.<br>Note: Evaluate $f$ at all critical points and endpoints. |
| *Critical Point | The point $(c,f(c))$ where $f'(c)=0$ or undefined is called critical point. |
| Stationary point | Any point where f is neither increasing nor decreasing but $f'(c)=0$ is called stationary point |
| Point of Inflexion | The point $(c,f(c))$ is called point of inflexion if $f'(x)$ does not change sign at $x=c$. At this point $f''(c)=0$ or $f''(c)$ does not exist. |



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## Concavity

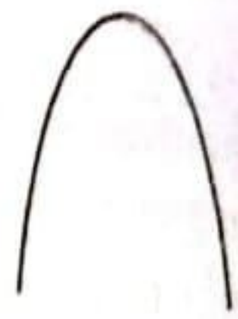
**Concave up:**

A portion of a graph that is cupped upward is called concave up. The graph of a function is concave upward on a open interval $(a,b)$ where $f''(x)>0$.



**Concave down:**

A portion of a graph that is cupped downward is called concave down. The graph of a function is concave downward on a open interval $(a,b)$ where $f''(x)<0$.



* The only domain points where extreme values exist are critical or end points.

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# 3 CHAPTER INTEGRATION

## Differential

The term differential is used in calculus (branch of Mathematics) to refer to an infinitesimal (infinitely small) change in some varying quantities called variable.
For example the differential $dx$ represents infinitely small change in variable "$x$"

## Anti-Derivative

Let $f(x)$ be a given function. If there is a differentiable function $F(x)$ such that $\frac{d}{dx}(F(x)) = f(x)$

Then $F(x)$ is called anti-derivative or primitive of $f(x)$

## Indefinite Integral

If $F(x)$ is an anti-derivative of $f(x)$, then the expression $F(x)+c$ is called indefinite integral of $f(x)$.

i.e. $\int f(x)\,dx = F(x)+c$

The process of finding anti-derivative is called **integration**.

**Note:**

- If $y = \int f(x)dx$ , then $\frac{dy}{dx} = f(x)$.
- Since the derivative of a constant is zero, all indefinite integrals differ by an arbitrary constant.

$$\frac{d}{dx}\int f(x)\,dx = f(x)$$

## Rules of integration

**Rule 1:** $\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + c, \quad n \neq -1$

**Rule 2:** $\int [f(x)]^{-1} f'(x)\,dx = \ln f(x) + c, \quad f(x) > 0$

## Some General Formulae of Integration

(i) $\int x^n dx = \frac{x^{n+1}}{n+1} + c, \quad n \neq -1$

(ii) $\int \frac{1}{x} dx = \ln x + c$

(iii) $\int cf(x)dx = c\int f(x)dx$ , $c$ is a constant

(iv) $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

Where $f(x)$ and $g(x)$ are continuous function



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## Integration of Trigonometric Functions

(i) $\int \sin x dx = -\cos x + c$, $\int \sin mx dx = -\frac{\cos mx}{m} + c$

(ii) $\int \cos x dx = \sin x + c$, $\int \cos mx dx = \frac{\sin mx}{m} + c$

(iii) $\int \sec^2 x dx = \tan x + c$

(iv) $\int \operatorname{cosec}^2 x dx = -\cot x + c$

(v) $\int \sec x \tan x dx = \sec x + c$

(vi) $\int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + c$

(vii) $\int \tan x dx = \ln \sec x + c = -\ln \cos x + c$

(viii) $\int \cot x dx = \ln \sin x + c$

(ix) $\int \sec x dx = \ln|\sec x + \tan x| = \ln \tan\left(x + \frac{\pi}{2}\right) + c$

(x) $\int \operatorname{cosec} x dx = \ln|\operatorname{cosec} x - \cot x| = \ln \tan\left(\frac{x}{2}\right) + c$

**Some Useful Results:**

(i) $\int \tan^2 x dx = \tan x - x + c$

(ii) $\int \cot^2 x dx = -\cot x - x + c$

(iii) $\int \sin^2 x dx = \frac{x}{2} - \frac{\sin 2x}{4} = \frac{1}{2}(x - \sin x \cos x) + c$

(iv) $\int \cos^2 x dx = \frac{x}{2} + \frac{\sin 2x}{4} = \frac{1}{2}(x + \sin x \cos x) + c$

## Integration Involving Inverse Trigonometric Functions

(i) $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c$ OR $-\cos^{-1} x + c$

(ii) $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + c$ OR $-\operatorname{cosec} x^{-1} x + c$

(iii) $\int \frac{dx}{1+x^2} = \tan^{-1} x + c$ OR $-\cot^{-1} x + c$

(iv) $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

(v) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$

(vi) $\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c$



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## Integration of Hyperbolic Functions

(i) $\int \sinh x dx = \cosh x + c$

(ii) $\int \cosh x dx = \sinh x + c$

(iii) $\int \sec h^2 x dx = \tan hx + c$

(iv) $\int \text{cosech}^2\, x dx = -\coth x + c$

(v) $\int \tanh x \,\text{sech}\, x dx = -\text{sech}\, x + c$

(vi) $\int \text{cosech}\, x \coth x dx = -\cos echx + c$

## Integration Involving Inverse Hyperbolic Functions

(i) $\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1} x + c, \quad x \in R$

(ii) $\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1} x + c$

(iii) $\int \frac{dx}{1-x^2} = \begin{cases} \tanh^{-1} x + c, & |x| < 1 \\ \coth^{-1} x + c, & |x| > 1 \end{cases}$

(iv) $\int \frac{-1}{x\sqrt{1-x^2}} dx = \text{sech}^{-1} x + c, \quad 0 < x < 1$

(v) $\int \frac{-1}{|x|\sqrt{1+x^2}} dx = \text{cosech}^{-1} x + c, \quad x \in R - \{0\}$

## Some Integrals Including Logarithms

(i) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + c = -\frac{1}{a} \coth^{-1} \left( \frac{x}{a} \right) + c, |x| < a$

(ii) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + c = \frac{1}{a} \tanh^{-1} \left( \frac{x}{a} \right) + c, |x| < a$

(iii) $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln \left| x + \sqrt{x^2+a^2} \right| + c = \sinh^{-1} \frac{x}{a} + c$

(iv) $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln \left| x + \sqrt{x^2-a^2} \right| + c = \cosh^{-1} \frac{x}{a} + c$

(v) $\int \sqrt{x^2+a^2}\, dx = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2} \ln \left| x + \sqrt{x^2+a^2} \right| + c$

(vi) $\int \sqrt{x^2-a^2}\, dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2} \ln \left| x + \sqrt{x^2-a^2} \right| + c$



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## Integration of Exponential Functions

(i) $\int e^{f(x)}dx = \frac{e^{f(x)}}{f'(x)} + c$

(ii) $\int e^{x}dx = e^{x} + c$

(iii) $\int a^{f(x)}dx = \frac{a^{f(x)}}{f'(x).\ln a} + c$

(iv) $\int a^{x}dx = \frac{a^{x}}{\ln a} + c$ , a > 0

## Integration by Substitution

Sometimes it is possible to convert an integral into a standard form or to an easy integral by a suitable change of a variable.

| EXPRESSION INVOLVING | SUITABLE SUBSTITUTION |
|---|---|
| $\sqrt{a^2 - x^2}$ | $x = a\sin\theta$ |
| $\sqrt{x^2 - a^2}$ | $x = a\sec\theta$ |
| $\sqrt{a^2 + x^2}$ | $x = a\tan\theta$ |
| $\sqrt{x+a}$ or $\sqrt{x-a}$ | $\sqrt{x+a} = t$ or $\sqrt{x-a} = t$ |
| $\sqrt{2ax - x^2}$ | $x - a = a\sin\theta$ |
| $\sqrt{2ax + x^2}$ | $x + a = a\sec\theta$ |

## Integration by Parts

The process of integration of the product of two functions is known as integration by parts. For example, if $u$ and $v$ are two functions of $x$, then

$$\int (uv)\,dx = u\int v dx - \int \left(\frac{du}{dx}\int v dx\right)dx$$

This is called integration by parts.

If the two functions are of different type, we can choose the first function as the one whose initial comes first in the word **"ILATE"** where

I – Inverse trigonometric function
L – Logarithmic function
A – Algebraic function
T – Trigonometric function
E – Exponential function

## Deduction From Integration by Parts

(i) $\int udv = uv - \int vdu$

(ii) $\int e^{x}\left(f(x) + f'(x)\right)dx = e^{x}f(x) + c$

(iii) $\int e^{ax}\left(af(x) + f'(x)\right)dx = e^{ax}f(x) + c$



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(iv) $\int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2} + c = \frac{1}{\sqrt{a^2+b^2}} e^{ax}\cos\left(bx - \tan^{-1}\frac{b}{a}\right) + c$

(v) $\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2} + c = \frac{1}{\sqrt{a^2+b^2}} e^{ax}\sin\left(bx - \tan^{-1}\frac{b}{a}\right) + c$

## Definite Integrals

### Fundamental Theorem of Calculus

If $f$ is continuous on $[a,b]$ and $\phi'(x) = f(x)$, that is, $\phi(x)$ is any anti derivative of $f$ on $[a,b]$, then $\int_a^b f(x)dx = \phi(b) - \phi(a)$

### Properties of Definite Integral

(i) $\int_a^b f(x)dx = \left|F(x)\right|_a^b = F(b) - F(a)$

(ii) $\int_a^a f(x)dx = 0$

(iii) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

(iv) $\int_a^b cf(x)dx = c\int_a^b f(x)dx$, ( $c$ is a constant)

(v) $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

(vi) $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$ ,where $a < b < c$

(vii) $\int_a^b f(x)dx = \int_a^b f(t)dt$ (independent of variable)

(viii) $\int_{-a}^a f(x)dx = 0$, if $f(x)$ is an odd function.

(ix) $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$, if $f(x)$ is an even function.

(x) $\frac{d}{dx}\int_a^x f(t)dt = f(x)$

(xi) $\int_a^b f(x)dx = \int_{a-c}^{b-c} f(x+c)dx = \int_{a+c}^{b+c} f(x-c)dx$

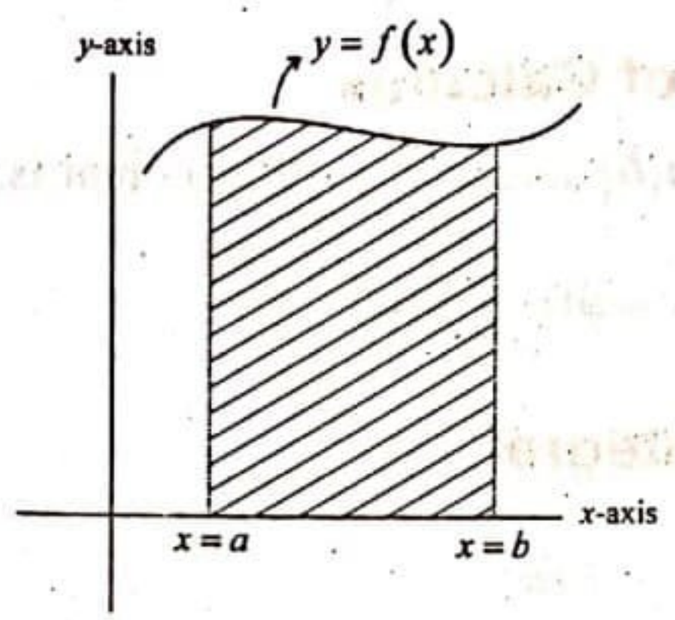
(xii) If $f(x) \geq g(x)$ in $[a,b]$ then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$

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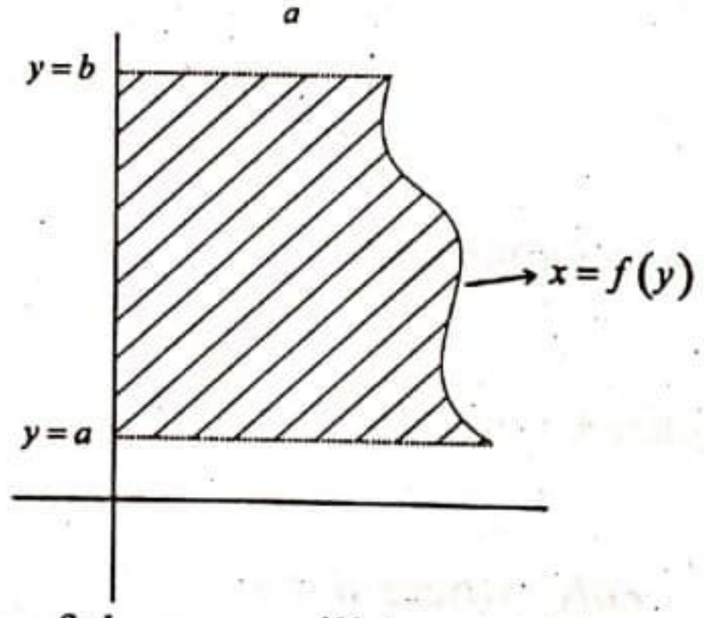
## Applications of Integration

**(i) Area Under Curves and The Axes:**

**(a)** If '$A_1$' is the area bounded by curve $y = f(x)$, **x-axis** and between the coordinates $x = a$, $x = b$ then $A_1 = \int_a^b y\,dx$



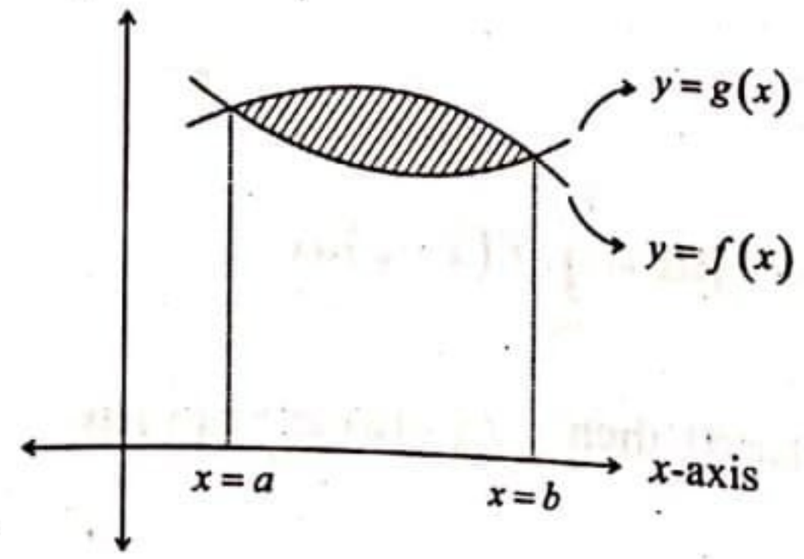

**(b)** If '$A_2$' is the area bounded by the curve $y = f(x)$, **y-axis** and between the Coordinates $y = a$, $y = b$, then $A_2 = \int_a^b x\,dy$




Only the numerical value of the area will be considered (i.e. if there comes negative sign with the area after calculating neglect it.)

**(ii) Area between Two Curves:**

Let $f(x)$ and $g(x)$ be two curves such that $f(x) \geq g(x)$ for every $x \in [a,b]$, then the area of the region bounded by the graphs of $f(x)$ and $g(x)$ and the coordinates from $x = a$ to $x = b$ for $a \leq x \leq b$ is given by $A = \left| \int_a^b (f(x) - g(x))\,dx \right|$






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# Differential Equations

An equation involving an independent, dependent variable and the derivative of the dependent variable is called a Differential Equation.

## Order of Differential Equation

The order of the differential equation is the order of the highest derivative that occurs in the differential equation.

## Degree of Differential Equation

The degree of the differential equation is the degree of the highest order derivative that appears in the differential equation.

**Note:**

The dependent variable and its derivatives should be expressed in a form free from radicals and fractions.

## Solution of Differential Equation

A relation between the dependent and the independent variables of the differential equation i.e. a function of the form $y = f(x)+c$ which satisfies a given differential equation is called its solution.

## General Solution

If the solution of differential equation of $n^{th}$ order contains $n$ arbitrary constants, then it is called the general solution.

## Particular Solution

A solution obtained by giving particular values to arbitrary constants in the general solution is called a particular solution.

## Initial Conditions

The arbitrary constants involving in the solution of differential equations can be determined by the given conditions, such conditions are called initial value conditions.

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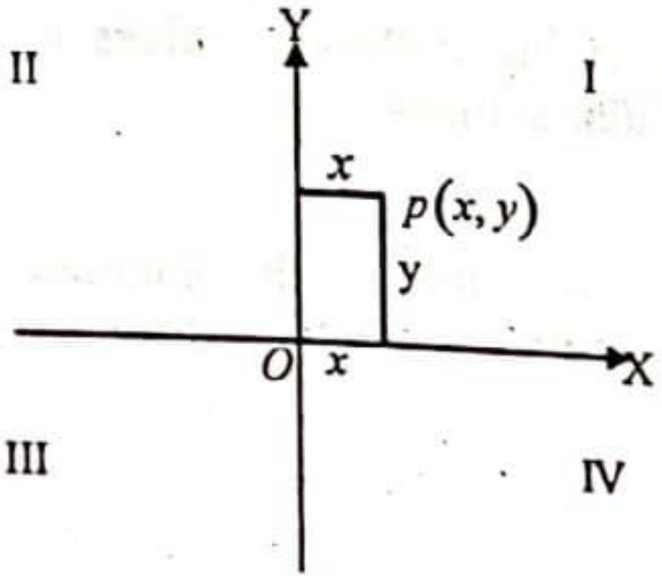
# CHAPTER 4 INTRODUCTION TO ANALYTIC GEOMETRY

## Coordinate System

The system of coordinates use in analytic geometry is defined by the familiar Cartesian plane used in algebra and trigonometry. As shown in the figure the plane is divided into four quadrants by the two perpendicular lines ( $X-axis$ and $Y-axis$ ) intersecting at the origin $O$.

They are defined as;

| | |
|---|---|
| **Quadrant I** | All points $(x, y)$ with $x>0, y>0$ |
| **Quadrant II** | All points $(x, y)$ with $x<0, y>0$ |
| **Quadrant III** | All points $(x, y)$ with $x<0, y<0$ |
| **Quadrant IV** | All points $(x, y)$ with $x>0, y<0$ |




## Some Terms Related To Cartesian plane

| | |
|---|---|
| **Abscissa** | The distance of a point from $Y-axis$ is called *x-coordinate* or *abscissa*. |
| **Ordinate** | The distance of a point from $X-axis$ is called *y-coordinate* or *ordinate*. |
| **Cartesian / Rectangular Coordinates** | The two distances taken together and enclosed in parentheses $(x, y)$ are called Cartesian or rectangular coordinates. |



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## Definitions and Formulae

| | |
|---|---|
| **The Distance Formula** | The length or the distance of the line segment joining the points $A(x_1, y_1)$ & $B(x_2, y_2)$ is given by; $d = \|AB\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ |
| **Distance of a Point from Origin** | Let $A(x_1, y_1)$ be any point then its distance from the origin $O(0,0)$ is given by; $\|OA\| = \sqrt{x_1^2 + y_1^2}$ |
| **Centroid of a Triangle** | The point of intersection of medians of a triangle is called centroid. Let $A(x_1, y_1)$ , $B(x_2, y_2)$ and $C(x_3, y_3)$ are the vertices of a triangle then , centroid of triangle $= \left( \frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right)$ |
| **In-centre of a Triangle** | The point of intersection of angle bisectors of a triangle is called in-centre. Let $A(x_1, y_1)$ , $B(x_2, y_2)$ and $C(x_3, y_3)$ are the vertices of a triangle and $a$,$b$ and $c$ are the lengths of the sides opposite to the vertices $A$ ,$B$ and $C$ respectively ,then in-centre of the triangle $= \left( \frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c} \right)$ |

**Remember:**
Centroid and in-center of a triangle always lies within triangle.

## The Ratio formula

| | |
|---|---|
| **Internal Division** | Let $A(x_1, y_1)$ & $B(x_2, y_2)$ be the two points in a plane. The coordinates of the point dividing the line segment $AB$ in the ratio $k_1 : k_2$ internally are $\left( \frac{k_1 x_2 + k_2 x_1}{k_1 + k_2}, \frac{k_1 y_2 + k_2 y_1}{k_1 + k_2} \right)$ |
| **External Division** | Let $A(x_1, y_1)$ & $B(x_2, y_2)$ be the two points in a plane. The coordinates of the point dividing the line segment $AB$ in the ratio $k_1 : k_2$ externally are $\left( \frac{k_1 x_2 - k_2 x_1}{k_1 - k_2}, \frac{k_1 y_2 - k_2 y_1}{k_1 - k_2} \right)$. |
| **The Midpoint** | If $k_1 : k_2 = 1:1$ ,then the point of division becomes the mid-point of $A(x_1, y_1)$ and $B(x_2, y_2)$ ,so the coordinates of the point are $\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$. |

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## Theorems*

(i) The centroid of a triangle is a point that divides each median in the ratio 2:1.
(ii) The medians of a triangle are concurrent.
(iii) The angle bisectors of a triangle are concurrent.
(iv) The ratio formula is valid in which ever quadrants the points lie.
(v) The angle bisector divides the opposite side in the same ratio as the other two sides have.

## Transformations

Changing the position of the reference axis in a coordinate system by **translation, rotation** or both, usually to simplify the equation of a curve, is called transformation.

| TRANSLATION OF AXIS | ROTATION OF AXIS |
|---|---|
| Let $p(x,y)$ be a point in $xy$-coordinate system. If the origin $O(0,0)$ is translated to a new point $O'(h,k)$, then point $p(x,y)$ are translated to $p'(X,Y)$, where<br>$X = x-h$<br>$Y = y-k$<br>• Remember that in translation of axis ,origin is shifted to another point but the axes remain parallel to the old axes. | The $xy$-coordinate axes are rotated about the origin through an angle $\theta\left(0<\theta<\frac{\pi}{2}\right)$. Then point $p(x,y)$ are transformed to $p'(X,Y)$, where<br>$X = x\cos\theta + y\sin\theta$<br>$Y = y\cos\theta - x\sin\theta$<br>• In rotation of axes, origin remains at its position, while the axes are shifted to new axes. |

## Some Terms Related To a Line

| | |
|---|---|
| **Inclination** | Inclination is the angle $(\alpha), 0^o < \alpha < 180^o$ measured anti clockwise, that a line makes with the positive $X-axis$. |
| **Inclinations of some standard straight lines** | (a) Inclination of $X-axis$ or any line parallel to $X-axis$ is $0^\circ$.<br>(b) Inclination of $Y-axis$ or any line parallel to $Y-axis$ is $90^\circ$. |
| **Slope/Gradient** | The tangent of inclination of a line is called slope, denoted by $m$. *i.e.* $slope = m = \tan\alpha$. |
| **Slope of a line segment** | If $A(x_1,y_1)$ & $B(x_2,y_2)$ are any two points on a line ,then the slope of the line segment is defined as; $m = \frac{y_2-y_1}{x_2-x_1}$ |
| **Slopes of some standard straight lines** | (a) The slope of $X-axis$ or any line parallel to $X-axis$ is 0.<br>(b) The slope of $Y-axis$ or any line parallel to $Y-axis$ is undefined. |
| **Theorems related to slope** | Let $m_1$ & $m_2$ be the slopes of two straight lines $l_1$ & $l_2$ respectively, then;<br>(i) $l_1 \parallel l_2$ iff $m_1 = m_2$.<br>(ii) $l_1 \perp l_2$ iff $m_1.m_2 = -1$.<br>(iii)If slope of AB = slope of BC<br>Then the points A,B and C are collinear. |

* The proofs of all the theorems are beyond the scope of this book.



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## Equations of Straight Lines Under Different Conditions

| | |
|---|---|
| Equation of X – axis | $y=0$ |
| Equation of Y – axis | $x=0$ |
| Equation of a line parallel to X – axis | $y=k$ , $k$ is any constant not equal to zero. |
| Equation of line parallel to Y – axis | $x=h$ , $h$ is any constant not equal to zero. |
| Slope –Intercept form | Let $m$ be the slope and $c$ be the y-intercept of a non-vertical line then the equation of the line is $y=mx+c$ |
| Equation of Line having slope m and passing through Origin | $y=mx$ |
| Point-Slope Form | Equation of a non-vertical line having slope $m$ and passing through a point $P(x_1,y_1)$ is $y-y_1=m(x-x_1)$. |
| Two Points Form | Equation of a line passing through two points $A(x_1,y_1)$ & $B(x_2,y_2)$ is. (i) $\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$ (ii) $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}=0$ |
| Intercepts Form | Equation of line making intercepts $a,b$ on $X$-and $Y$-axes respectively is $\frac{x}{a}+\frac{y}{b}=1$ |
| Normal/Perpendicular Form | Equation of line with perpendicular length $p$ from origin and perpendicular making an angle $\alpha$ with positive X-axis is $x\cos\alpha+y\sin\alpha=p$. |

## General Form of a Straight Line

$ax+by+c=0$ is the general form of a straight line, where $a,b,c$ are any real numbers and $a,b$ cannot zero simultaneously.

Different parameters of different forms of the straight line can be expressed in terms of $a$, $b$, $c$ of the general form of a line as follow;

- Slope , $m=-\frac{a}{b}$
- $x$-intercept $=-\frac{c}{a}$
- $y$-intercept $=-\frac{c}{b}$
- Distance from origin $=p=\frac{|c|}{\sqrt{a^2+b^2}}$

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## Position of a Point with Respect to a Line

(a) The point $P(x_1, y_1)$ lies above, on or below the line $ax+by+c=0$, according as $ax_1+by_1+c \gtreqless 0\ (b>0)$.

(b) The point $P(x_1, y_1)$ is above or below the line $ax+by+c=0$ if $ax_1+by_1+c$ and $b$ have the same or opposite signs.

(c) The $P(x_1, y_1)$ and the origin are

(i) On the same side of the line according as $ax_1+by_1+c$ and $c$ have same signs.

(ii) On the opposite side of the line according as $ax_1+by_1+c$ and $c$ have opposite signs.

## Two and Three Straight Lines

Two lines $a_1x+b_1y+c_1=0$ & $a_2x+b_2y+c_2=0$ are said to be;

(i) Parallel iff $\frac{a_1}{a_2}=\frac{b_1}{b_2}$.

(ii) Perpendicular iff $a_1a_2+b_1b_2=0$.

(iii) Identical iff $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$.

(iv) Intersects each other iff $a_1b_2-a_2b_1\neq 0$

## The Point of Intersection of Two Straight Lines

Let $a_1x+b_1y+c_1=0$ & $a_2x+b_2y+c_2=0$ be two non-parallel and coplanar lines then their point of intersection is;

$$\left(\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}, \frac{a_2c_1-a_1c_2}{a_1b_2-a_2b_1}\right), a_1b_2-a_2b_1\neq 0$$

## Condition of Concurrency of Three Lines

Three non-parallel lines

$l_1: a_1x+b_1y+c_1=0$

$l_2: a_2x+b_2y+c_2=0$ are concurrent if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0$

$l_3: a_3x+b_3y+c_3=0$

This is necessary and sufficient condition of concurrency.

## Distance of a Point $P(x_1, y_1)$ From a Line

The distance $d$ from the point $P(x_1, y_1)$ to the line $l: ax+by+c=0$ is given by; $d=\frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$



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## Distance Between Two Parallel Lines

The distance between two parallel lines $\begin{matrix} l_1 : ax+by+c=0 \\ l_2 : ax+by+c'=0 \end{matrix}$ is given as; $d=\dfrac{|c-c'|}{\sqrt{a^2+b^2}}$

## Area of a Triangular Region

Let $A(x_1,y_1)$ , $B(x_2,y_2)$ and $C(x_3,y_3)$ be the vertices of a triangular region $ABC$ then the area of the triangular region is given as; $\Delta=\dfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$.

## Trapezium

A quadrilateral having two parallel sides and two non parallel sides is called Trapezium.

Area of Trapezium = $\dfrac{1}{2}$ (sum of parallel sides) (distance between parallel sides)

## Angle Between Two Lines

Let $l_1$ and $l_2$ be two non-vertical lines such that they are not perpendicular to each other. If $m_1$ and $m_2$ are the slopes of $l_1$ and $l_2$ respectively, then the angle from $l_1$ to $l_2$ is given by; $\tan\theta=\dfrac{m_2-m_1}{1+m_1m_2}$ or $\theta=\tan^{-1}\dfrac{m_2-m_1}{1+m_1m_2}$,

## Acute Angle Between Two Straight Lines

The acute angle between two lines $\begin{matrix} l_1 : a_1x+b_1y+c_1=0 \\ l_2 : a_2x+b_2y+c_2=0 \end{matrix}$ is given $\tan\theta=\left|\dfrac{a_1b_2-a_2b_1}{a_1a_2+b_1b_2}\right|$

**Key Fact:**

Orthocenter and circumcenter change their positions with respect to shape of triangle.

| | |
|---|---|
| **Orthocenter** | The point of concurrency of altitudes of a triangle is called orthocentre. |
| **Circumcenter** | The point of concurrency of perpendicular bisectors of a triangle is called circumcentre. |

**Theorem:** Altitudes of a triangle are concurrent.

**Theorem:** Right bisectors of a triangle are concurrent.

**Theorem:** If the points $A$, $B$, $C$ are collinear then $\Delta=0$

## Family of Lines

Let $\ell_1 : a_1x+b_1y+c_1=0$ *and* $\ell_2 : a_2x+b_2y+c_2=0$ for non zero real h, the equation $(a_1x+b_1y+c_1)+h(a_2x+b_2y+c_2)=0$ represents family of lines.



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## Pair of Straight Lines

| | |
|---|---|
| **Homogeneous Equation** | An equation of the type $f(x,y)=0$ is called homogeneous equation of degree $n$ ($n$ is positive integer) if $f(kx,ky)=k^n f(x,y)$ |
| **General Homogeneous Equation of Second Degree in Two Variables** | $ax^2+2hxy+by^2=0$, where $a, h$ and $b$ are not simultaneously zero. |

**Theorem:** every homogeneous second degree equation $ax^2+2hxy+by^2=0$ represents a pair of lines through the origin. The lines are ;

**(i)** Real and distinct, if $h^2>ab$.

**(ii)** Real and coincident, if $h^2=ab$.

**(iii)** Imaginary, if $h^2<ab$.

Angle between the pair of lines represented by $ax^2+2hxy+by^2=0$ is given as;

$$\tan\theta=\frac{2\sqrt{h^2-ab}}{a+b}$$

**(i)** $\theta=0$ if $h^2-ab=0$. (Lines are coincident)

**(ii)** $\theta=\frac{\pi}{2}$ if $a+b=0$. (Lines are perpendicular/orthogonal)

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# 5 CHAPTER

# LINEAR INEQUALITIES AND LINEAR PROGRAMMING

## Inequalities

Inequalities are of four types:

| Inequality | Symbol |
|---|---|
| Greater than | $>$ |
| Less than | $<$ |
| Greater than or equal to | $\geq$ |
| Less than or equal to | $\leq$ |

## Linear Inequalities

Generally a linear inequality in two variables $x$ and $y$ can be one of the following four forms:

(i) $ax+by<c$

(ii) $ax+by\leq c$

(iii) $ax+by>c$

(iv) $ax+by\geq c$, where $a$, $b$ and $c$ are constants also $a$, $b$ cannot be zero simultaneously.

## Some Important Results Related To Linear Inequality In Two Variables

- $ax+by=c$ is called Associated or Corresponding Equation of any of the above four linear inequalities.
- Every inequality divides a plane into two half planes.
- A vertical line divides the plane into left and right half planes.
- A non-vertical line divides the plane into upper and lower half planes.
- If an inequality involves $<$ or $>$, the half plane is open.
- If an inequality involves $\leq$ or $\geq$, the half plane is closed.
- A point of a solution region where two of its boundary lines intersect is called a corner point or vertex.

## Procedure for Graphing a Linear Inequality in Two Variables

(i) Graph the corresponding equation $ax+by=c$ by using dashes line if the inequality involves the symbols $<$ or $>$ and a solid line is drawn if the inequality involves the symbols $\leq$ or $\geq$

(ii) The solution set is the half plane on one side of the boundary line. To determine which side, choose a test point (any point in the plane but not on the line) and check to see if the coordinates of the point satisfy the given inequality.

## Linear Programming

Linear programming is a mathematical technique for maximizing or minimizing a linear function. It is also called linear optimization.

## Some Terms Related to Linear Programming

### Problem Constraints

The system of linear inequalities involved in the problem concerned is called problem constraints.

### Non-negative Constraints

The variables used in the system of linear inequalities relating to the problems of everyday life are non-negative and are called non-negative constraints



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## Decision Variables

Non-negative constraints are also called decision variables.

## Feasible Region

A restricted region under certain constraints is called the feasible region.

## Feasible Solution

Each point of the feasible region is called feasible solution.

## Feasible Solution Set

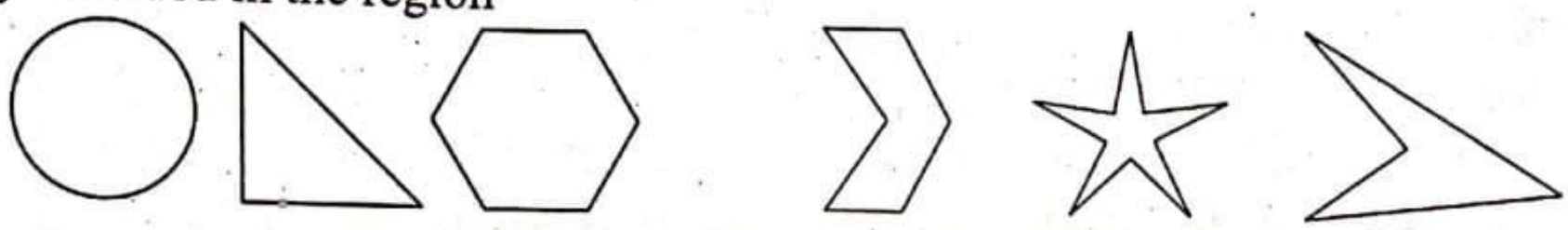
The set of all feasible solutions of a system of linear inequalities is called feasible solution set.

## Convex Region

If the line joining any two points of a region lies entirely within the region, then the region is called convex.

## Concave Region

A region is concave region if at least one line segment between two different points is not fully included in the region



Convex Regions

Concave Regions

## Bounded Region

A region which can be enclosed within a circle is known as bounded region.

## Unbounded Region

A region which cannot be enclosed within a circle is known as unbounded region.

## Objective Function

A function which is to be maximized or minimized but not both is called objective function.

## Optimal Solution

The feasible solution which maximizes or minimizes the objective function is called optimal solution.

## Theorem of Linear Programming

The theorem states that the maximum and minimum values of the objective function occurs at corner points of the feasible region.

## Procedure for Determining Optimal Solution

(i) Graph the solution set of linear inequality constraints to determine the feasible region.
(ii) Find the corner points of the feasible region.
(iii) Evaluate the objective function at each corner point to find the optimal solution.

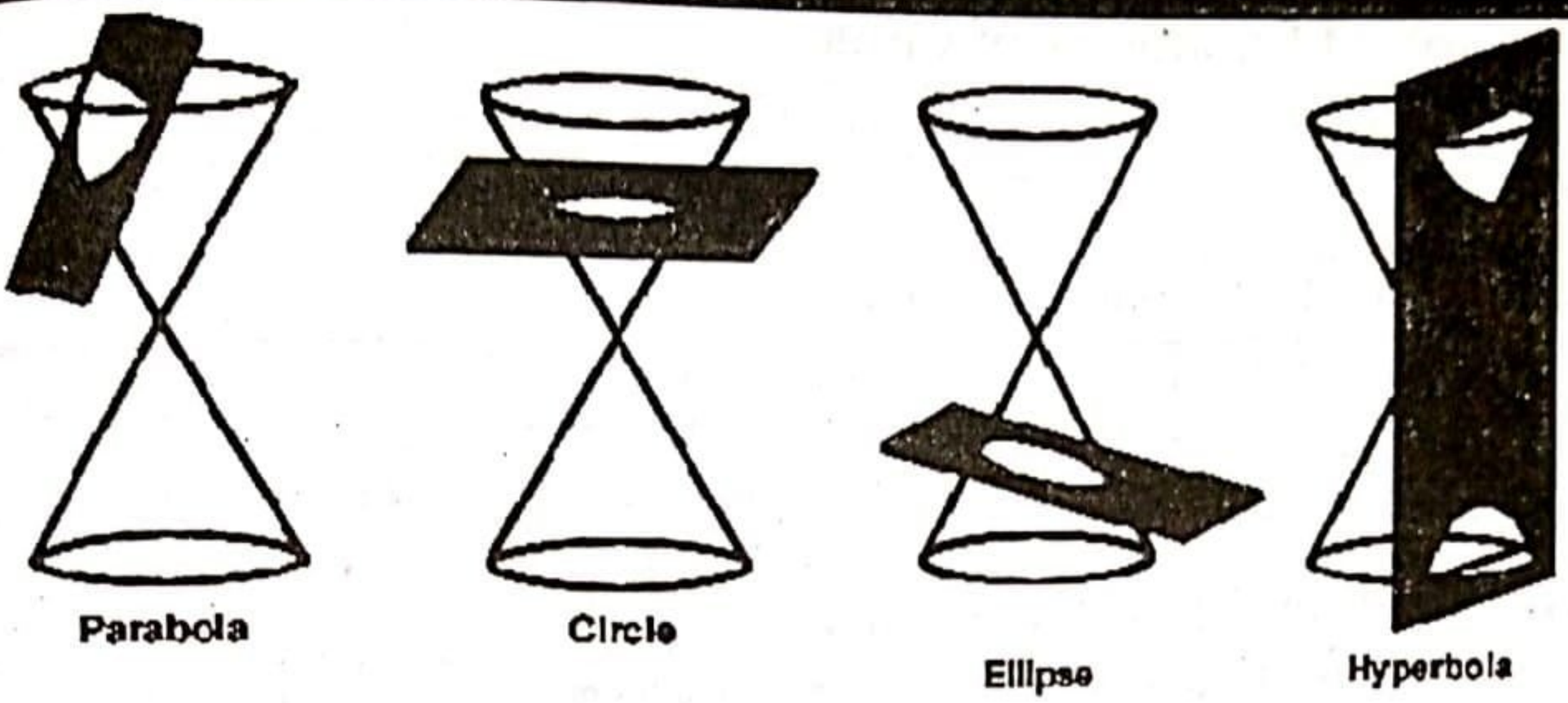
## Points to Be Remember

- There are infinite many feasible solutions in the feasible region.
- The point of intersection of two boundary lines is a corner point only if it lies in feasible region.
- If a linear program has an optimal solution, it must occur at a corner point of the feasible region.



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# 6 CHAPTER CONIC SECTION




## The Cone and Its Sections

| | |
|---|---|
| **Right Circular Cone** | The surface generated by straight lines passing through a point and ending on the circumference of the circle. The lines are called generators of the cone. |
| **Axis of The Cone** | The fixed line of the cone is called its axis. |
| **Elements of The Cone** | The possible positions of the generating line in its rotation about the axis are its elements. |
| **Vertex or Apex** | The common intersection point of all the cone's elements is the cone's vertex or apex. |
| **Nappes** | The two symmetrical parts of the generated surface on each side of the vertex are the cone's nappes. |
| **Conic Sections or Conics** | The curves that can be obtained by cutting a cone with a plane are called conic sections or simply conics. |
| **Circle** | When the intersecting plane cuts completely across one nappe at a right angle to the axis of the cone, the curve is a circle. |
| **Point Circle** | If the intersecting plane cuts across the vertex of a cone, the resulting curve is a point circle. |
| **Parabola** | When the intersecting plane is parallel to the generator of the form and passing through only one nappe, the resulting curve is parabola. |
| **Ellipse** | When the intersecting plane is slightly tilted and passing through only one nappe then the curve is an ellipse. |
| **Hyperbola** | When the intersecting plane cuts through both nappes parallel to the axis of the cone the resulting curve is hyperbola. |



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## Circle

A set of points such that distance of each point from a fixed point (**centre**) remains constant. The constant distance is called **radius** of the circle.

## Standard Form of Equations of Circle

| | |
|---|---|
| Centre $(h,k)$, radius $= r$ | $(x-h)^2+(y-k)^2=r^2$ |
| Centre $(0,0)$, radius $= r$ | $x^2+y^2=r^2$ |
| Centre, $(0,0)$ radius = 1 (Unit circle) | $x^2+y^2=1$ |
| Centre $(0,0)$, radius = 0 (Point circle) | $x^2+y^2=0$ |
| If end points of the diameter of the circle are $A(x_1,y_1)$ & $B(x_2,y_2)$ | $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$ |

## Parametric Form of Equations of Circle

| Standard form | Parametric form |
|---|---|
| $(x-h)^2+(y-k)^2=r^2$ | $x=h+r\cos\theta, y=k+r\sin\theta$ |
| $x^2+y^2=r^2$ | $x=r\cos\theta, y=r\sin\theta$ |

## General Form of a Circle

The general equation of the circle is $x^2+y^2+2gx+2fy+c=0$.

## Properties and Results From General Equation

(i) The general equation of the circle involves three constants, i.e. $g, f$ & $c$.

(ii) The general equation of a circle is a second degree equation in which coefficients of $x^2$ and $y^2$ are 1.

(iii) The general equation does not contain the term involving the product $xy$.

| | |
|---|---|
| **Center of the Circle** | $(-g,-f)$ *i.e.* $\left(-\frac{\text{coefficient of } x}{2}, -\frac{\text{coefficient of } y}{2}\right)$ |
| **Radius of the Circle** | (i) $r=\sqrt{g^2+f^2-c} > 0$ Real circle.<br>(ii) $r=\sqrt{g^2+f^2-c}=0$ , point circle.<br>(iii) If $g^2+f^2-c<0$ ,imaginary circle. |

## Tangents and Normals

| | |
|---|---|
| Tangent | A straight line that touches the curve at a point without cutting the curve. |
| Normal | A straight line perpendicular to the curve at the point of tangency. |
| Equation of tangent to the circle $x^2+y^2=a^2$ at $P(x_1,y_1)$. | $xx_1+yy_1=a^2$ |
| Equation of normal to the circle $x^2+y^2=a^2$ at $P(x_1,y_1)$. | $xy_1-yx_1=0$ |
| Equation of tangent to the circle | $xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$ |

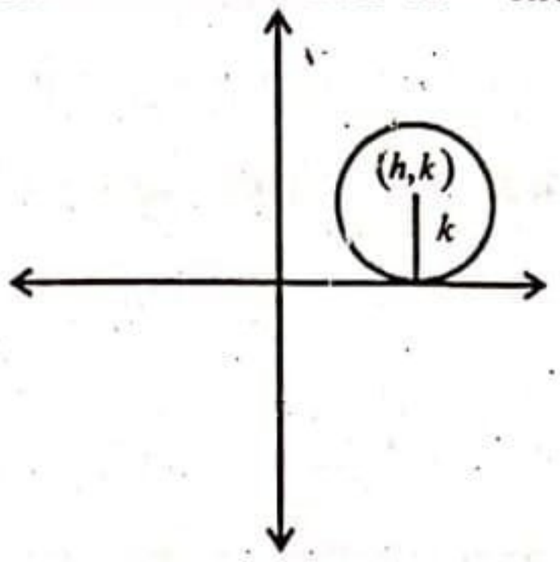


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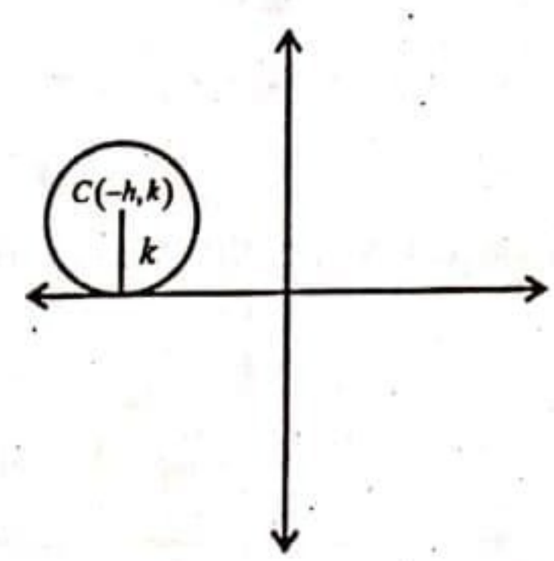
| | |
|---|---|
| $x^2+y^2+2gx+2fy+c=0$ at $P(x_1,y_1)$. | |
| Equation of normal to the circle $x^2+y^2+2gx+2fy+c=0$ at $P(x_1,y_1)$. | $(x-x_1)(y_1+f)-(y-y_1)(x_1+g)=0$ |

## Equation of a Circle in Some Special Cases

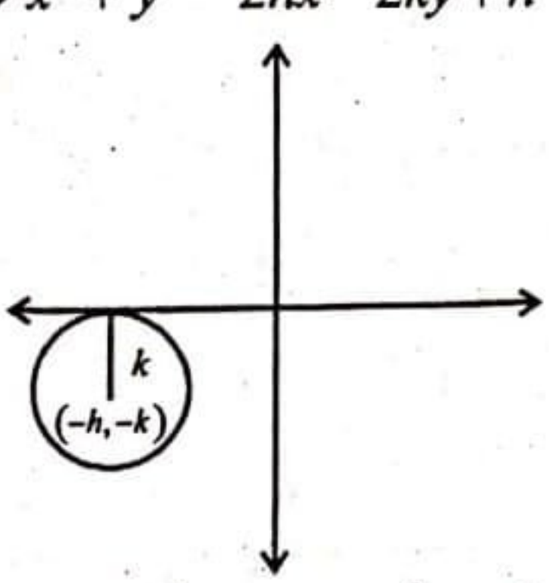
(i) If the circle touches $x-axis$ then its equation is :



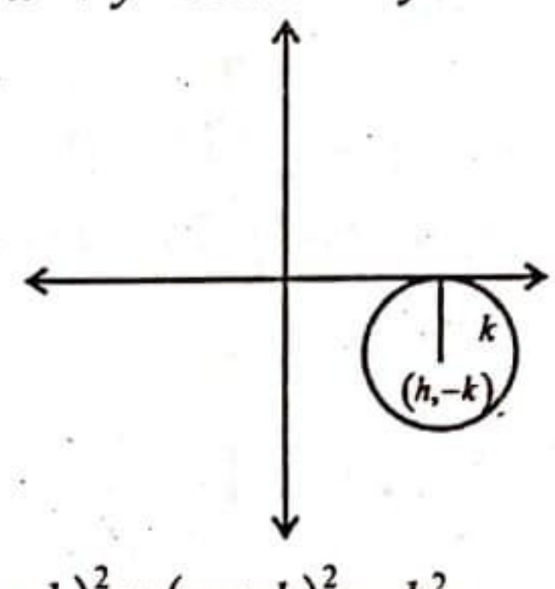

$$(x-h)^2+(y-k)^2=k^2$$
$$\Rightarrow x^2+y^2-2hx-2ky+h^2=0$$




$$(x+h)^2+(y-k)^2=k^2$$
$$\Rightarrow x^2+y^2+2hx-2ky+h^2=0$$

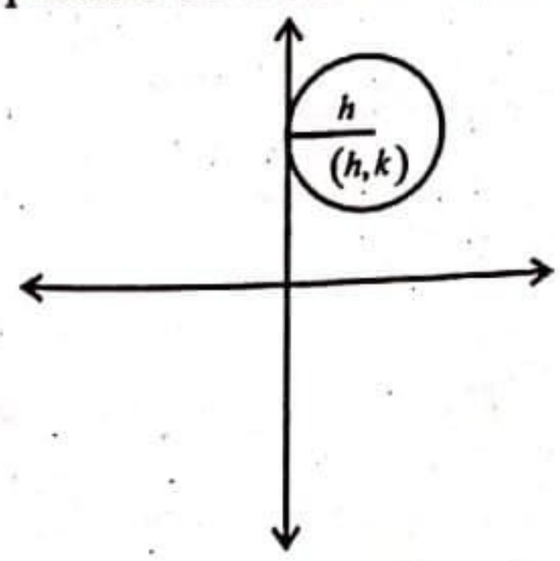



$$(x+h)^2+(y+k)^2=k^2$$
$$\Rightarrow x^2+y^2+2hx+2ky+h^2=0$$

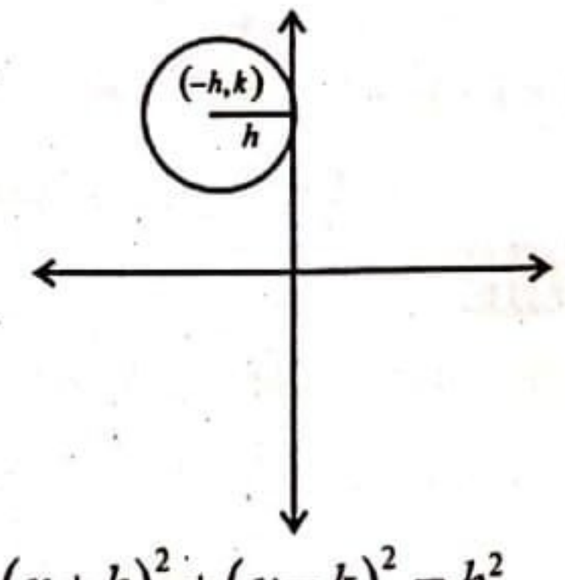



$$(x-h)^2+(y+k)^2=k^2$$
$$\Rightarrow x^2+y^2-2hx+2ky+h^2=0$$

(ii) Equation of a circle touches $y-axis$ then its equation is.



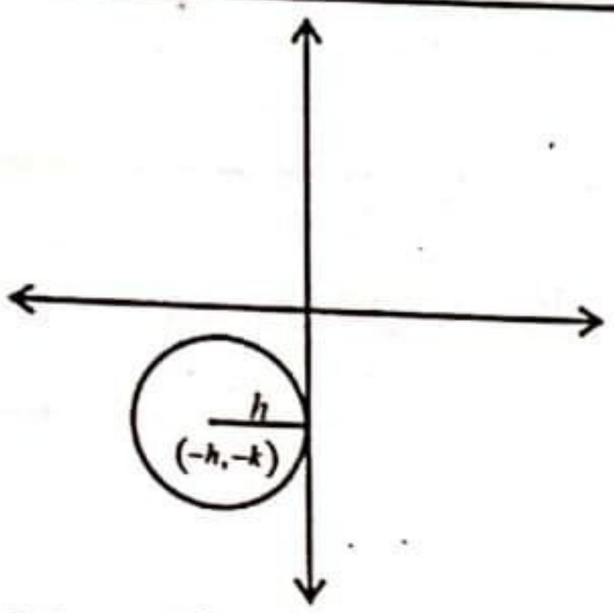

$$(x-h)^2+(y-k)^2=h^2$$
$$\Rightarrow x^2+y^2-2hx-2ky+k^2=0$$




$$(x+h)^2+(y-k)^2=h^2$$
$$\Rightarrow x^2+y^2+2hx-2ky+k^2=0$$

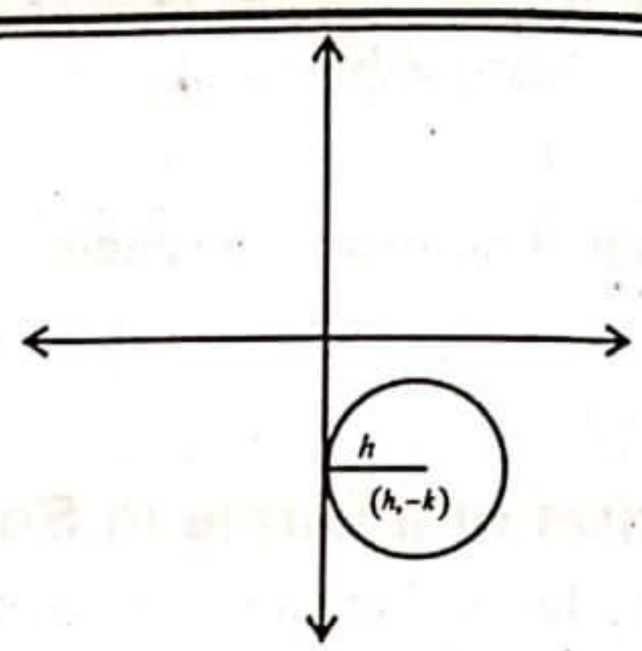


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$$(x+h)^2+(y+k)^2=h^2$$

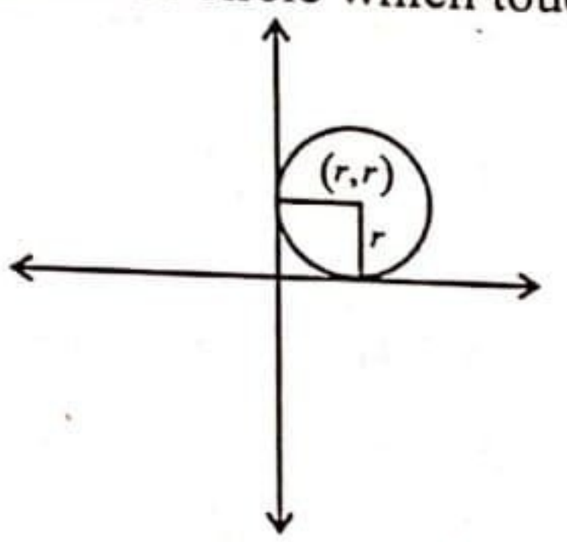
$$\Rightarrow x^2+y^2+2hx+2ky+k^2=0$$




$$(x-h)^2+(y+k)^2=h^2$$

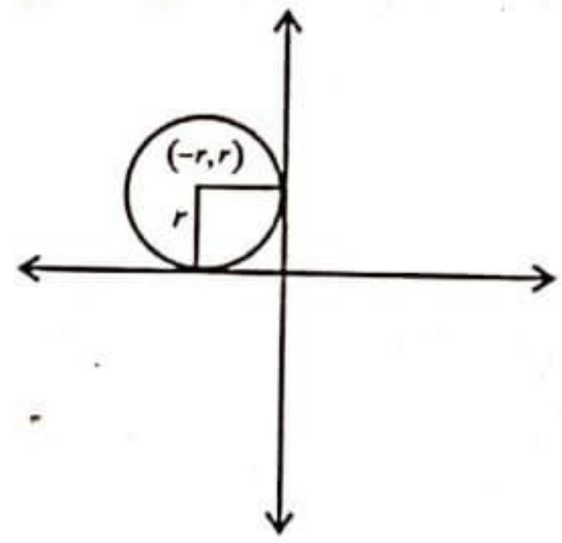
$$\Rightarrow x^2+y^2-2hx+2ky+k^2=0$$

**(iii)** Equation of circle which touches both the axes is



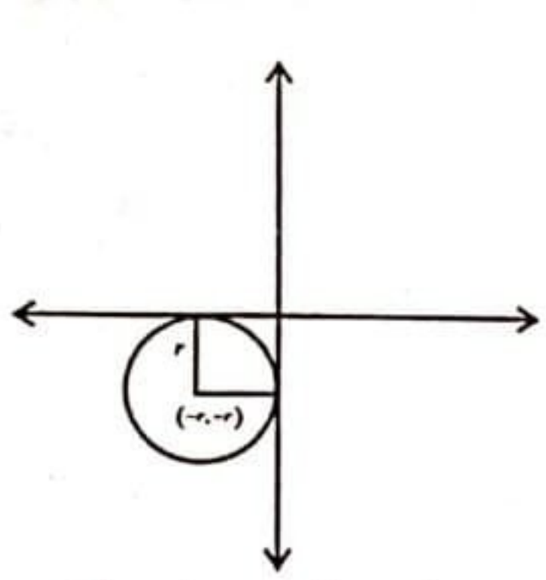

$$(x-r)^2+(y-r)^2=r^2$$

$$\Rightarrow x^2+y^2-2rx-2ry+r^2=0$$



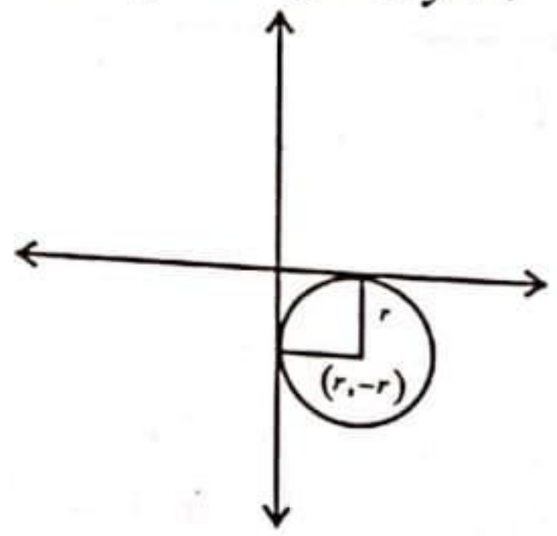

$$(x+r)^2+(y-r)^2=r^2$$

$$\Rightarrow x^2+y^2+2rx-2ry+r^2=0$$




$$(x+r)^2+(y+r)^2=r^2$$

$$\Rightarrow x^2+y^2+2rx+2ry+r^2=0$$




$$(x-r)^2+(y+r)^2=r^2$$

$$\Rightarrow x^2+y^2-2rx+2ry+r^2=0$$

## Theorems

**(i)** The point $P(x_1,y_1)$ lies outside, on or inside the circle $x^2+y^2+2gx+2fy+c=0$ according as $x_1^2+y_1^2+2gx_1+2fy_1+c \gtreqless 0$.

**(ii)** Two tangents can be drawn to a circle from any point outside the circle.

**(iii)** Let $P(x_1,y_1)$ be a point outside the circle $x^2+y^2+2gx+2fy+c=0$, then length of either tangent drawn from $P(x_1,y_1)$ to the circle $=\sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}$



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## Condition of Tangency

(i) The line $y = mx + c$ touches the circle $x^2 + y^2 = a^2$ if, $c = \pm a\sqrt{1+m^2}$

## Theorems

The line $y = mx + c$ intersects the circle $x^2 + y^2 = a^2$ in at the most two points. These points are

(a) Real and distinct if, $a^2\left(1+m^2\right) - c^2 > 0$

(b) Real and coincident if, $a^2\left(1+m^2\right) - c^2 = 0$

(c) Imaginary if, $a^2\left(1+m^2\right) - c^2 < 0$

## Important Properties of a Circle*

(i) Perpendicular dropped from the center of a circle on a chord bisects the chord

(ii) The perpendicular bisector of any chord of a circle passes through the center of the circle.

(iii) The line joining the center of a circle to the midpoint of a chord is perpendicular to the chord.

(iv) Congruent chords of a circle are equidistant from its centre.

(v) Measure of the central angle of a minor arc is double the measure of the angle subtended in the corresponding major arc.

(vi) An angle in a semi-circle is a right angle.

(vii) The tangent to a circle at any point of the circle is perpendicular to the radial segment at that point

(viii) The perpendicular at the outer end of a radial segment is tangent to the circle.

(ix) Normal lines of a circle pass through the centre of the circle.

(x) Mid point of the hypotenuse of a right triangle is the circumcentre of the triangle

(xi) Perpendicular dropped from a point of a circle on a diameter is a mean proportional between the segments into which it divides the diameter.

## Contact of Two Circles

Let $r_1$, $r_2$ be the radii of two circles and $C_1$, $C_2$ be their centers, then

| | |
|---|---|
| Circles Touch Externally | $\lvert C_1C_2\rvert = r_1 + r_2$ |
| Circles Touch Internally | $\lvert C_1C_2\rvert = \lvert r_1 - r_2\rvert$ |
| Circles Intersect in Two Real Distinct Points | $\lvert r_1 - r_2\rvert < \lvert C_1C_2\rvert < r_1 + r_2$ |
| Circles do not Touch | (i) $\lvert C_1C_2\rvert > r_1 + r_2$<br>(ii) $\lvert C_1C_2\rvert < \lvert r_1 - r_2\rvert$ |

## Parabola

A set of points in a plane such that the distance of each point form a fixed point (Focus, F) is equal to its distance from a fixed straight line (Directrix, L).

* Analytic proofs of these properties are beyond the scope of this book.

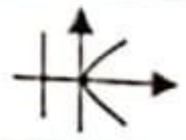
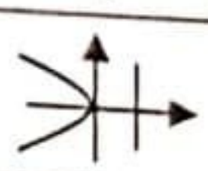
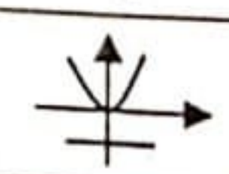
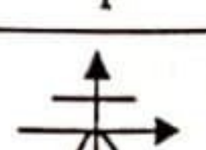


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## Terms Related to Parabola

| | |
|---|---|
| **Axis of Parabola** | The line through focus and perpendicular to directrix is called axis of parabola. |
| **Vertex** | The mid point of the perpendicular line joining focus and directrix is called vertex (or turning point) of parabola. |
| **Focal Chord** | A chord of the parabola through focus is called focal chord. |
| **Latus Rectum** | The focal chord perpendicular to the axis of the parabola is called latus rectum of the parabola. |
| **Eccentricity** | The ratio of the distance of any point on the parabola to its distance from directrix. |

## Main Facts about Standard Forms of Parabola

| **Equation** | $y^2 = 4ax$ | $y^2 = -4ax$ | $x^2 = 4ay$ | $x^2 = -4ay$ |
|---|---|---|---|---|
| **Focus** | $(a,0)$ | $(-a,0)$ | $(0,a)$ | $(0,-a)$ |
| **Vertex** | $(0,0)$ | $(0,0)$ | $(0,0)$ | $(0,0)$ |
| **Length of Latus Rectum** | $4a$ | $4a$ | $4a$ | $4a$ |
| **Equation of Latus Rectum** | $x = a$ | $x = -a$ | $y = a$ | $y = -a$ |
| **Equation of Directrix** | $x = -a$ | $x = a$ | $y = -a$ | $y = a$ |
| **Axis** | $y = 0$ | $y = 0$ | $x = 0$ | $x = 0$ |
| **Eccentricity** | 1 | 1 | 1 | 1 |
| **Graph** |  |  |  |  |

## Parametric Form of Equations of Parabola

| STANDARD FORM | PARAMETRIC FORM |
|---|---|
| $y^2 = 4ax$ | $x = at^2, y = 2at$ |
| $y^2 = -4ax$ | $x = -at^2, y = 2at$ |
| $x^2 = 4ay$ | $x = 2at, y = at^2$ |
| $x^2 = -4ay$ | $x = 2at, y = -at^2$ |

## Theorems on Parabola

(i) The point on the parabola closest to the focus is the vertex.

(ii) The ordinate at any point P of the parabola is a mean proportional between the length of the latus rectum and the abscissa of P.

(iii) The tangent at any point P of a parabola makes equal angles with the line PF and the line through P parallel to the axis of parabola, F being focus.

## Tangents and Normals

| | |
|---|---|
| **Equation of tangent to parabola $y^2 = 4ax$ is** | $yy_1 = 2a(x + x_1)$ |
| **Equation of normal to parabola $y^2 = 4ax$ is** | $y - y_1 = -\frac{y_1}{2a}(x - x_1)$ |



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## Condition for y = mx + c Tangent to Parabola

| EQUATION | CONDITION |
|---|---|
| $y^2 = 4ax$ | $c = \frac{a}{m}\ \ m \neq 0$ |
| $y^2 = -4ax$ | $c = -\frac{a}{m}\ \ m \neq 0$ |
| $x^2 = 4ay$ | $c = -am^2$ |
| $x^2 = -4ay$ | $c = am^2$ |

The line $y = mx + c$ will intersect parabola $y^2 = 4ax$ at

(i) Two distinct points if $c < \frac{a}{m}$

(ii) A single point if $c = \frac{a}{m}$

(iii) No point at all if $c > \frac{a}{m}$

**Note:**

The line $y = mx + c$ will be normal to parabola $y^2 = 4ax$ if $c = -2am - am^3$

## Ellipse

A set of points such that the distance of each point from a fixed point (Focus, F) bears a constant ratio (Eccentricity, $0 < e < 1$) to its perpendicular distance from a fixed straight line (Directrix, L).

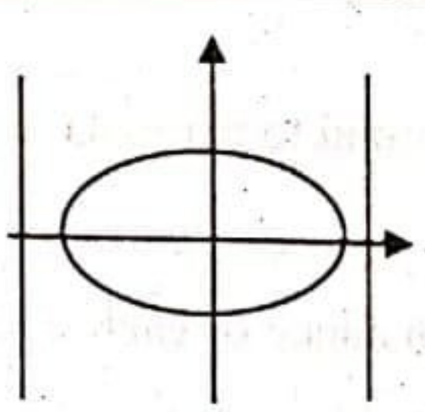
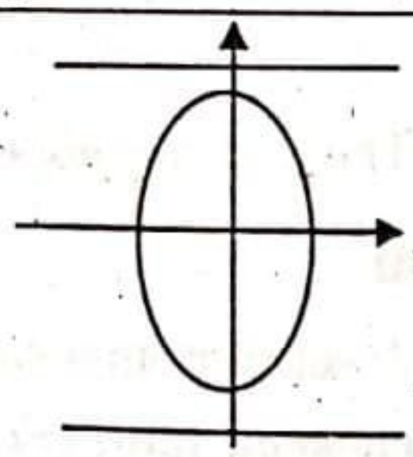
## Terms Related To Standard Ellipse

| | |
|---|---|
| Vertices | The points on the standard ellipse where it crosses the major axis. |
| Covertices | The points on the standard ellipse where it crosses the minor axis. |
| Center | The midpoint of the line joining vertices (or covertices or Foci). |
| Major axis | The line joining vertices is called major axis. |
| Minor axis | The line joining co vertices is called minor axis. |
| Latera Recta | The chords perpendicular to major axis passes through foci are called latera recta. |
| Eccentricity | The ratio of the distance of any point on the ellipse from the focus to its distance from the directrix. |



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## Main Facts about Standard Forms of Ellipse

| Equation | $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, a>b$ <br> $c^2=a^2-b^2$ | $\frac{y^2}{a^2}+\frac{x^2}{b^2}=1, a>b$ <br> $c^2=a^2-b^2$ |
|---|---|---|
| Foci | $(\pm c,0)$ | $(0,\pm c)$ |
| Eccentricity | $0<e=\frac{c}{a}<1$ | $0<e=\frac{c}{a}<1$ |
| Vertices | $(\pm a,0)$ | $(0,\pm a)$ |
| Co vertices | $(0,\pm b)$ | $(\pm b,0)$ |
| Center | $(0,0)$ | $(0,0)$ |
| Length of Major Axis | $2a$ | $2a$ |
| Equation of Major Axis | $y=0$ | $x=0$ |
| Length of Minor Axis | $2b$ | $2b$ |
| Equation of Minor Axis | $x=0$ | $y=0$ |
| Length of Latus Rectum | $\frac{2b^2}{a}$ | $\frac{2b^2}{a}$ |
| Equation of Directrices | $x=\pm\frac{c}{e^2}$ | $y=\pm\frac{c}{e^2}$ |
| Graph |  |  |

## Parametric Form of Equations of Ellipse

| STANDARD FORM | PARAMETRIC FORM |
|---|---|
| $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ | $x=a\cos\theta$ <br> $y=b\sin\theta$ |
| $\frac{y^2}{a^2}+\frac{x^2}{b^2}=1$ | $x=b\cos\theta$ <br> $y=a\sin\theta$ |
| $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ | $x=h+a\cos\theta$ <br> $y=k+b\sin\theta$ |

## Theorems

(i) The sum of the focal distances of any point on an ellipse is equal to length of major axis (2a).

(ii) The distance between centre and any focus of the ellipse is denoted by $c$, and is given as; $c=\sqrt{a^2-b^2}$.

(iii) The distance between foci $=2c$.



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## Tangents and Normals

| | |
|---|---|
| Equation of tangent to Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ | is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$ |
| Equation of normal to Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ | is $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$ |

## Condition for y = mx + c Tangent to Ellipse

| EQUATION | CONDITION |
|---|---|
| $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ | $c^2=a^2m^2+b^2$ |
| $\frac{y^2}{a^2}+\frac{x^2}{b^2}=1$ | $c^2=a^2+m^2b^2$ |

**Theorem**

The line $y=mx+c$ will intersect Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at

(i) Two distinct points if $c^2<a^2m^2+b^2$

(ii) A single point if $c^2=a^2m^2+b^2$

(iii) No point at all if $c^2>a^2m^2+b^2$

## Hyperbola

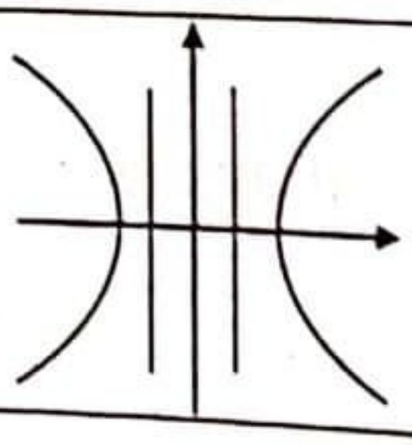
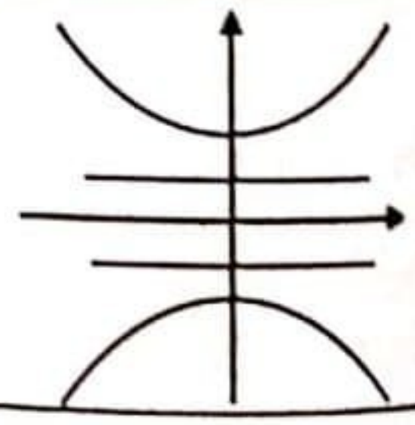
A set of points such that the distance of each point from a fixed point (focus, $F$) bears a constant ratio (Eccentricity, $e>1$) to its perpendicular distance from a fixed straight line (directrix, $L$).

## Terms Related To Standard Hyperbola

| | |
|---|---|
| Vertices | The points on the standard hyperbola where it crosses the *x-axis*. |
| Center | The midpoint of the line joining vertices (or co vertices). |
| Transverse Axis | The line joining vertices is called transverse axis. |
| Conjugate Axis | The line joining co vertices are called conjugate axis. |
| Latera Recta | The chords perpendicular to major axis passes through foci are called latera recta. |
| Eccentricity | The ratio of the distance of any point on the ellipse from the focus to its distance from the directrix. |
| Asymptotes | In general, an asymptote is a line that approaches a curve but never touches.<br>Every hyperbola is associated with two lines called asymptotes. Their point of intersection is centre of the hyperbola.<br>(a) Slope of asymptotes $=\pm\frac{b}{a}$, hyperbola along *x–axis*.<br>(b) Slope of asymptotes $=\pm\frac{a}{b}$, hyperbola along *y–axis*. |

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## Main Facts about Standard Forms of Hyperbola

| Equation | $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1, c^2=a^2+b^2$ either $a>b, b>a$ or $a=b$ | $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1,\ c^2=a^2+b^2$ either $a>b, b>a$ or $a=b$ |
|---|---|---|
| Foci | $(\pm c, 0)$ | $(0, \pm c)$ |
| Eccentricity | $e=\frac{c}{a}>1$ | $e=\frac{c}{a}>1$ |
| Vertices | $(\pm a, 0)$ | $(0, \pm a)$ |
| Center | $(0,0)$ | $(0,0)$ |
| Length of Transverse Axis | $2a$ | $2a$ |
| Equation of Transverse Axis | $y=0$ | $x=0$ |
| Length of Conjugate Axis | $2b$ | $2b$ |
| Equation of Conjugate Axis | $x=0$ | $y=0$ |
| Length of Latus Rectum | $\frac{2b^2}{a}$ | $\frac{2b^2}{a}$ |
| Equation of Directrices | $x=\pm\frac{c}{e^2}$ | $y=\pm\frac{c}{e^2}$ |
| Equation of Asymptotes | $y=\pm\frac{b}{a}x$ | $y=\pm\frac{a}{b}x$ |
| Graph |  |  |

## Rectangular Hyperbola

A hyperbola is said to be rectangular if its conjugate and transverse axis are equal. *i.e* $2a=2b \Rightarrow a=b$.

In case of rectangular hyperbola the equation of hyperbola is reduced to $x^2-y^2=a^2$.

## Properties of Rectangular Hyperbola

(i) The asymptotes of rectangular hyperbola are perpendicular to each other and equation of asymptotes of rectangular hyperbola are $y=\pm x$

(ii) The Eccentricity of rectangular hyperbola is: $e=\sqrt{1+\frac{b^2}{a^2}}=\sqrt{1+\frac{a^2}{a^2}}=\sqrt{2}$

(iii) Equation for rectangular hyperbola referred to its asymptotes as co-ordinate axis is $xy=c^2$ where $c^2=\frac{a^2}{2}$ and its parametric equation are $x=ct, y=\frac{c}{t}$



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## Parametric Form of Equations of Hyperbola

| STANDARD FORM | PARAMETRIC FORM |
|---|---|
| $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ | $x=a\sec\theta \quad x=a\cosh\theta$<br>$y=b\tan\theta' \quad y=b\sinh\theta$ |
| $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ | $x=b\tan\theta$<br>$y=a\sec\theta$ |
| $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ | $x=h+a\sec\theta$<br>$y=k+b\tan\theta$ |

## Theorems

(i) The difference of the focal distances of any point on the hyperbola is equal to the length of transverse axis ($2a$).

(ii) The distance between centre and any focus of the hyperbola is denoted by $c$ and is given as; $c=\sqrt{a^2+b^2}$.

(iii) The distance between the foci $= 2c$

(iv) Angle between the asymptotes of hyperbola is $2\tan^{-1}\left(\frac{b}{a}\right)$.

(v) The product of distances from the foci to any tangent to the hyperbola is $b^2$.

## Tangents and Normals

| | |
|---|---|
| **Equation of tangent to hyperbola** $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ | **is** $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$ |
| **Equation of normal to hyperbola** $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ | **is** $\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2$ |

## Condition for y = mx + c Tangent to Hyperbola

| EQUATION | CONDITION |
|---|---|
| $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ | $c^2=a^2m^2-b^2$ |
| $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ | $c^2=a^2-m^2b^2$ |

## For Equation of Tangent to a Curve

| | |
|---|---|
| Generally to find an equation of tangent at a point $P(x_1,y_1)$, make the replacements in the given equation of the curve. | Replace<br>$x^2$ by $xx_1$<br>$y^2$ by $yy_1$<br>$xy$ by $\frac{xy_1+yx_1}{2}$<br>$x$ by $\frac{x+x_1}{2}$<br>$y$ by $\frac{y+y_1}{2}$ |



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## Identification of a Conic From the General Equation

The general equation of a conic $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents;

(i) A pair of straight lines, if $\Delta=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0$

(ii) An ellipse or circle if $h^2-ab<0, \Delta\neq 0$

(iii) A parabola if $h^2-ab=0, \Delta\neq 0$

(iv) A hyperbola if $h^2-ab>0, \Delta\neq 0$

**Note:**

Centre of general conic is $\left(\frac{hf-bg}{ab-h^2}, \frac{gh-af}{ab-h^2}\right)$

## LOCUS

A locus is a set of points satisfying given condition(s).

| 2-DIMENSIONAL | 3-DIMENSIONAL |
|---|---|
| **(i)** The set of all points in a plane at a constant distance from a fixed point is a **circle.** | **(i)** The set of all points in space at a constant distance from a fixed point is a **sphere.** |
| **(ii)** The set of all points in a plane at a constant distance from a given straight line is a pair of straight lines parallel to the given line. | **(ii)** The set of all points in space at a constant distance from a given straight line is a **cylinder.** |
| **(iii)** The set of all points in a plane such that the sum of their distances from two fixed points is a constant is an **ellipse.** | **(iii)** In space an ellipse is changed into an **ellipsoid.** |
| **(iv)** The set of all points in a plane such that the difference of their distances from two fixed points is a constant is a **hyperbola.** | **(iv)** In space hyperbola is shifted to a **hyperboloid.** |
| **(v)** The set of all points such that their distances from a fixed point are equal to their distances from a fixed straight line is a **parabola.** | **(v)** In space parabola is called **paraboloid.** |
| **(vi)** The locus of a point in a plane that is equidistant from two fixed points is the perpendicular bisector (mediator) of the straight line joining those two points. | |
| **(vii)** The locus of a point in a plane equidistant from two intersecting straight lines is pair of straight lines bisects the angles between two given straight lines. | |

## Transformations

| | |
|---|---|
| **Translation of Axis** | To transform the given equation referred to the new origin $O'(h,k)$ put $x=X+h$ , $y=Y+k$ in the given equation of the curve. |
| **Rotation of Axis** | To find the new equation when axis are rotated through an angle $\theta$, put $x=X\cos\theta-Y\sin\theta$, $y=X\sin\theta+Y\cos\theta$ in the given equation. |
| **Elimination of the *xy*-Term** | To remove the xy-term from a second degree equation ,determine the value of $\theta$ ,such that $\tan 2\theta=\frac{2h}{a-b}$ |



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# CHAPTER 7 VECTORS

As we know, in **mathematics, physics** and **engineering** two important quantities are **scalars and vectors.**

| PHYSICAL QUANTITY | DESCRIPTION | EXAMPLES |
|---|---|---|
| Scalar Quantity /Scalar | The physical quantities, which have magnitude, only are called scalars. | Distance, speed, mass, length, time, volume, temperature and energy |
| Vector Quantity/Vector | The physical quantities that have magnitude, direction and obey addition laws are called vectors. | Displacement, velocity, acceleration, weight,force, momentum,electric and magnetic fields |

## Geometrical Interpretation of Vector

(a) Geometrically, a vector is represented by a directed line segment $\overrightarrow{AB}$ with $A$ as its initial point and $B$ as its terminal point.

(b) Usually a vector is denoted by an arrow written as $\overrightarrow{AB}$.

(c) A vector can also be represented by a bold face symbol like $\boldsymbol{v}$ or in underline form as $\underline{v}$.

## Some Important Terms Related To Vectors

| | |
|---|---|
| Magnitude / Length / Norm | The magnitude/length/norm of a vector $\overrightarrow{AB}$ or $\underline{v}$, is its absolute value and is denoted as $\lvert\overrightarrow{AB}\rvert$ or $AB$ or $\lvert\underline{v}\rvert$. |
| Support | The line of infinite length of which the vector is a part, is called its support. |
| Sense | The sense (direction) of a vector is from its initial point to its terminal point. |

## Some Types of Vectors

| NAME | DEFINITION | REPRESENTATION |
|---|---|---|
| Zero Vector / Null Vector | A vector whose initial and terminal points coincide. | A zero vector is represented by $\overline{O}$. |
| Proper Vector | A non zero vector is called proper vector. | $\vec{a}$ is proper vector iff $\lvert\vec{a}\rvert \neq 0$. |
| Unit Vector | A vector whose magnitude is 1 is called unit vector. | A unit vector in the direction of $\vec{a}$ is denoted by $\hat{a}$ and is given as $\hat{a} = \frac{\vec{a}}{\lvert\vec{a}\rvert}$. |

Unit vector is dimensionless physical quantity



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| | |
|---|---|
| **Fundamental Unit Vectors** | The unit vectors along $x$-, $y$- and $z$-axes are called fundamental unit vectors. $\underline{i}=[1,0,0]$, $\underline{j}=[0,1,0]$ & $\underline{k}=[0,0,1]$ are unit vectors along $x$-, $y$- and $z$-axes respectively. |
| **Co-Initial Vectors** | Two (more than two) vectors are called co-initial if they have the same initial point. $\overrightarrow{AC}$ , $\overrightarrow{AB}$ , $\overrightarrow{AD}$ are co-initial vectors. |
| **Negative of a Vector** | The vector which has the same magnitude as the vector $\vec{a}$ but has the direction opposite to that of $\vec{a}$. The negative of $\vec{a}$ is denoted by $-\vec{a}$. |
| **Position Vector** | The vector $\overrightarrow{OP}$, whose initial point $O$ is the origin and whose terminal point is $P$, is called position vector of point P. |
| **Co-Planar Vectors** | Two or more vectors are called co-planar if either they lie in the same plane or parallel to the same plane. |
| **Equal Vectors** | Two vectors are said to be equal if they have,<br>**(i)** The same length<br>**(ii)** Same or parallel support<br>**(iii)** The same sense |

| | |
|---|---|
| **Collinear or Parallel Vectors** | Two or more vectors are said to be collinear if they have either the same or parallel supports. |
| **Like And Unlike Vectors** | Two or more vectors are said to be **like vectors** when they have the same direction irrespective of their magnitudes and vectors having opposite directions are called **unlike vectors**. |
| **Reciprocal Vectors** | Two vectors are reciprocal of each other if they have same direction but their magnitudes are reciprocals of each other. |
| **Free Vector** | A vector, in which the initial point is not specified *i.e.* there is no restriction to choose its origin, is called a free vector. |
| **Localized Vector** | A vector with a fixed initial point is called a localized vector. |

## Addition of Vectors

Addition of vectors can be explained by the following laws;

| | |
|---|---|
| **Triangle Law of Vectors** | If two vectors are represented in magnitude and direction by the two sides of a triangle, taken in order then their sum (resultant) is represented in magnitude and direction by the third side of the triangle taken in opposite order. |
| **Parallelogram• Law of Vectors** | If two vectors are represented in magnitude and direction by two adjacent sides of a parallelogram, then their sum (resultant) is represented in magnitude and direction by the diagonal of the parallelogram with initial point of intersection of the two sides. |
| **Polygon Law of Vectors** | If a finite number of vectors are represented in magnitude and direction by the sides of an open polygon, taken in order, then their resultant is represented in magnitude and direction by the closing side of polygon, taken in the opposite order. |

• Aristotle used this law to describe the combine[illegible] of two forces.



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## Multiplication of a Vector by a Scalar

Let $\vec{a}$ be a vector and k a real number.

(i) If $k$ is a positive real number, then $k\vec{a}$ is defined to be a vector whose magnitude is $k$ time that of $\vec{a}$ and direction is the same as that of $\vec{a}$.

(ii) If $k$ is a negative real number, then $k\vec{a}$ is defined to be a vector whose magnitude is $k$ time the magnitude of $\vec{a}$ and direction opposite to that of $\vec{a}$.

(iii) If $k=0$, then $k\vec{a}$ is zero vector.

## Properties of Vector Addition

| | |
|---|---|
| Commutative | $\vec{a}+\vec{b}=\vec{b}+\vec{a}$ |
| Associative | $\vec{a}+\left(\vec{b}+\vec{c}\right)=\left(\vec{a}+\vec{b}\right)+\vec{c}$ |
| Difference of Two Vectors | $\vec{a}-\vec{b}=\vec{a}+\left(-\vec{b}\right)$ |
| Distributive Laws | (i) If $k$ is any real number and $\vec{a},\vec{b}$ are any two vectors, then $k\left(\vec{a}+\vec{b}\right)=k\vec{a}+k\vec{b}$.<br>(ii) If $\vec{a}$ is any vector and $m$, $n$ are real numbers, then $\left(m+n\right)\vec{a}=m\vec{a}+n\vec{a}$<br>(iii) $m\left(n\vec{a}\right)=n\left(m\vec{a}\right)$ |

## Vectors in Plane

- The set of all ordered pairs [x, y] of real numbers is called set of vectors in $R^2$.
- x and y are components of vector.
- $\underline{u}=\left[x,y\right]=x\underline{i}+y\underline{i}$
- $\left|\underline{u}\right|=\sqrt{x^2+y^2}$

## Vectors in Space

- The set of all ordered triples [x, y, z] of real numbers is called set of vectors in $R^3$.
- x, y and z are component of vector.
- $\underline{v}=\left[x,y,z\right]=x\underline{i}+y\underline{i}+z\underline{k}$
- $\left|\underline{v}\right|=\sqrt{x^2+y^2+z^2}$

## Distance Between Two Points in Space

Distance between two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in space is given by

$$\left|\overline{PQ}\right|=\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2+\left(z_2-z_1\right)^2}$$



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## Direction Ratios, Direction Angles and Direction Cosines of a Vector:

Let $\underline{r} = \overline{OP} = x\underline{i} + y\underline{j} + z\underline{k}$ be a non-zero vector, then any triplet of numbers proportional to $x$, $y$ and $z$ are called **direction ratios** of $\underline{r}$.

Let $\alpha, \beta, \gamma$ are angles formed by $\underline{r} = \overline{OP} = x\underline{i} + y\underline{j} + z\underline{k}$ with $x$-,$y$- and $z$-axes respectively such that $0 \le \alpha \le \pi$, $0 \le \beta \le \pi$ & $0 \le \gamma \le \pi$, are called direction angles of $\underline{r}$.

The numbers $\cos\alpha, \cos\beta$ and $\cos\gamma$ are called direction cosines of the vector $\underline{r}$. $\cos\alpha, \cos\beta$ and $\cos\gamma$ are also denoted by $l$, $m$ and $n$ respectively.

**Note:**

(i) $\cos\alpha = \frac{x}{|\underline{r}|}$, $\cos\beta = \frac{y}{|\underline{r}|}$ and $\cos\gamma = \frac{z}{|\underline{r}|}$.

(ii) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ i.e. $l^2 + m^2 + n^2 = 1$

## The Ratio Formula

(i) Let $A$ & $B$ be two points whose position vectors are $\underline{a}$ *and* $\underline{b}$ respectively. If a point $P$ divides $AB$ in ratio $p:q$, then the position vector $\underline{r}$ of $P$ is given as; $\underline{r} = \frac{q\underline{a} + p\underline{b}}{p+q}$.

(ii) Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be any two points in space. The coordinate of point C which divides $\overrightarrow{AB}$ in the ratio $m_1 : m_2$ are $\left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}, \frac{m_1z_2 + m_2z_1}{m_1 + m_2}\right)$

(iii) If $\frac{m_1}{m_2}$ is negative then the point C divides $\overrightarrow{AB}$ externally.

## Centroid

Point of intersection of all three medians in a triangle is called centroid.

If a triangle having three vertices $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ then centroid is given by

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$$

## Scalar Product of Two Vectors

| Scalar Product / Dot Product/Inner Product | For any two vectors $\vec{a}$ and $\vec{b}$ $\vec{a}.\vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\theta$, where $\theta$ is the angle between the vectors $\vec{a}$ and $\vec{b}$. |
|---|---|

If $\theta$ is acute then $\vec{a}.\vec{b}$ is positive and if $\theta$ is obtuse then $\vec{a}.\vec{b}$ is negative.



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| | |
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| **Scalar Product in Terms of Components** | If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ & $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ , then $\vec{a}.\vec{b}=a_1b_1+a_2b_2+a_3b_3$ |
| **Special Cases** | **(i)** $\hat{i}.\hat{i}=\hat{j}.\hat{j}=\hat{k}.\hat{k}=1$<br>**(ii)** $\hat{i}.\hat{j}=\hat{j}.\hat{k}=\hat{k}.\hat{i}=0$<br>**(iii)** If $\vec{a}.\vec{b}=0$ ,then either $\lvert\vec{a}\rvert=0$ or $\lvert\vec{b}\rvert=0$ or $\theta=\frac{\pi}{2}$ |

## Properties of Scalar Product

| | |
|---|---|
| **Orthogonal Vectors (Perpendicular vectors)** | Two vectors $\vec{a}$ & $\vec{b}$ are called orthogonal iff their dot product is zero. i.e. $\vec{a}.\vec{b}=0$ Iff $\vec{a}\perp\vec{b}$. |
| **Parallel vectors** | If two vectors $\vec{a}$ & $\vec{b}$ are parallel then $\theta=0$, thus $\vec{a}.\vec{b}=\lvert\vec{a}\rvert\lvert\vec{b}\rvert$. |
| **Cosine of the Angle Between Vectors** | The angle between the vectors $\vec{a}$ & $\vec{b}$ can be determined by using the definition of the scalar product; $\cos\theta=\frac{\vec{a}.\vec{b}}{\lvert\vec{a}\rvert\lvert\vec{b}\rvert},0\le\theta\le\pi$ |
| **Commutative Property** | Scalar product is commutative *i.e.* $\vec{a}.\vec{b}=\vec{b}.\vec{a}$ |
| **Associative Property** | The scalar product is associative with respect to a scalar $k$. $(k\vec{a}).\vec{b}=k(\vec{a}.\vec{b})=\vec{a}.(k\vec{b})$ |
| **Distributive Property** | Scalar product is distributive over addition, *i.e.* $\vec{a}.(\vec{b}+\vec{c})=\vec{a}.\vec{b}+\vec{a}.\vec{c}$ |

## Applications of Scalar Product

| | |
|---|---|
| **Projection of One Vector Upon Another Vector** | Let $\vec{a}$ & $\vec{b}$ are any two vectors then ,<br>**(i)** The projection of $\vec{a}$ along $\vec{b}$ $=\vec{a}.\hat{b}=\frac{\vec{a}.\vec{b}}{\lvert\underline{b}\rvert}$<br>**(ii)** The projection of $\vec{b}$ along $\vec{a}$ $=\vec{b}.\hat{a}=\frac{\vec{b}.\vec{a}}{\lvert\underline{a}\rvert}$ |
| **Angle Between Two Vectors** | If $\vec{a}=x_1\underline{i}+y_1\underline{j}+z_1\underline{k}$ and $\vec{b}=x_2\underline{i}+y_2\underline{j}+z_2\underline{k}$ then $\cos\theta=\frac{x_1x_2+y_1y_2+z_1z_2}{\sqrt{x_1^2+y_1^2+z_1^2}\sqrt{x_2^2+y_2^2+z_2^2}}$ |
| **Work Done by a Force** | The work done by a force $\vec{F}$ when its points of application undergoes a displacement $\vec{d}$ is given by; $w=\vec{F}.\vec{d}$ |



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## Vector Product of Two Vectors

| | |
|---|---|
| **Vector Product / Cross Product** | The vector or cross product of two vectors $\vec{a}$ & $\vec{b}$ is defined as, $\vec{a}\times\vec{b}=\left\|\vec{a}\right\|\left\|\vec{b}\right\|\sin\theta\hat{n}$, where $\theta$ is the angle between $\vec{a}$ & $\vec{b}$ and $\hat{n}$ is the unit vector perpendicular to both $\vec{a}$ & $\vec{b}$ |
| **Vector Product In Terms of Components** | If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ & $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, then $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\a_1&a_2&a_3\\b_1&b_2&b_3\end{vmatrix}$ |
| **Special Cases** | (i) If $\vec{a}\times\vec{b}=\vec{0}$ then either $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$ or $\vec{a}\parallel\vec{b}$<br>(ii) $\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=\vec{0}$.<br>(iii) $\hat{i}\times\hat{j}=\hat{k},\hat{j}\times\hat{k}=\hat{i},\hat{k}\times\hat{i}=\hat{j}$.<br>(iv) $\hat{j}\times\hat{i}=-\hat{k},\hat{k}\times\hat{j}=-\hat{i},\hat{i}\times\hat{k}=-\hat{j}$ |

## Properties of Vector Product

| | |
|---|---|
| **Parallel Vectors** | If two vectors $\vec{a}$ & $\vec{b}$ are parallel then $\theta=\pi$ or $\theta=0$, thus $\vec{a}\times\vec{b}=0$. |
| **Sine of the Angle Between Vectors** | The angle between the vectors $\vec{a}$ & $\vec{b}$ can be determined by using the definition of the vector product; $\sin\theta=\dfrac{\left\|\vec{a}\times\vec{b}\right\|}{\left\|\vec{a}\right\|\left\|\vec{b}\right\|},0\le\theta\le\pi$ |
| **Commutative Property** | Cross product is not commutative *i.e.* $\vec{a}\times\vec{b}\neq\vec{b}\times\vec{a}$ but $\vec{a}\times\vec{b}=-\vec{b}\times\vec{a}$ |
| **Associative Property** | The cross product is associative with respect to scalar $m$. $m\left(\vec{a}\times\vec{b}\right)=\left(m\vec{a}\right)\times\vec{b}=\vec{a}\times\left(m\vec{b}\right)=\left(\vec{a}\times\vec{b}\right)m$ |
| **Distributive Property** | Vector product is distributive with respect to vector addition.<br>(i). $\vec{a}\times\left(\vec{b}+\vec{c}\right)=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$<br>(ii). $\left(\vec{b}\times\vec{c}\right)+\vec{a}=\vec{b}\times\vec{a}+\vec{c}\times\vec{a}$ |

## Applications of Vector Product

| | |
|---|---|
| **Geometrical Meaning** | Geometrically $\left\|\vec{a}\times\vec{b}\right\|$ represents the area of a parallelogram whose adjacent sides are represented by the vectors $\vec{a}$ & $\vec{b}$ |
| **Unit Vector ⊥ to Both $\vec{a}$ & $\vec{b}$** | $\dfrac{\vec{a}\times\vec{b}}{\left\|\vec{a}\times\vec{b}\right\|}$ gives a unit vector ⊥ to both $\vec{a}$ & $\vec{b}$ |
| **Area of a Triangle** | The area of triangle having two sides $\vec{a}$ & $\vec{b}$ is equal to $\frac{1}{2}\left\|\vec{a}\times\vec{b}\right\|$. |

If $\vec{a}$ and $\vec{b}$ are diagonals of parallelogram then area of parallelogram $=\frac{1}{2}\left|\vec{a}\times\vec{b}\right|$



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| | |
|---|---|
| **Moment of a Force** | Moment of a force $\vec{F}$ acting at a point A about any other point B is given by $\overrightarrow{BA}\times\vec{F}$. |

## Scalar Triple Product

| | |
|---|---|
| **Scalar Triple Product / Box Product** | Let $\vec{a},\vec{b},\vec{c}$ be three vectors, then $\vec{a}.\vec{b}\times\vec{c}$ is called scalar triple product. It is denoted by $\left[\vec{a}\,\vec{b}\,\vec{c}\right]$. |
| **Scalar Triple Product in Terms of Components** | If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ & $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$, then $\vec{a}.\vec{b}\times\vec{c}=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$ |

## Applications of Scalar Triple Product

| | |
|---|---|
| **Geometrical Meaning** | Geometrically $\vec{a}.\vec{b}\times\vec{c}$ represents the volume of parallelopiped having $\vec{a},\vec{b},\vec{c}$ as its adjacent sides. |
| **Volume of a Tetrahedron** | If $\vec{a},\vec{b},\vec{c}$ are the coterminous edges of a tetrahedron then, volume of tetrahedron $=\frac{1}{6}\left\lvert\vec{a}.\vec{b}\times\vec{c}\right\rvert$ |
| **Coplanar Vectors** | If $\vec{a},\vec{b},\vec{c}$ are coplanar then $\vec{a}.\vec{b}\times\vec{c}=0$. |

## Properties of Scalar Triple Product

(i) In scalar triple product the dot and cross are interchangeable. *i.e* $\vec{a}.\vec{b}\times\vec{c}=\vec{a}\times\vec{b}.\vec{c}=\left[\vec{a}\,\vec{b}\,\vec{c}\right]$

also $\left[\vec{a}\,\vec{b}\,\vec{c}\right]=\left[\vec{b}\,\vec{c}\,\vec{a}\right]=\left[\vec{c}\,\vec{a}\,\vec{b}\right]$

$\Rightarrow\vec{a}.\left(\vec{b}\times\vec{c}\right)+\vec{b}.\left(\vec{c}\times\vec{a}\right)+\vec{c}.\left(\vec{a}\times\vec{b}\right)=3\vec{a}.\left(\vec{b}\times\vec{c}\right)$

(ii) If any two vectors are same in scalar triple product then the product is zero. $\vec{a}.\left(\vec{b}\times\vec{a}\right)=0$

(iii) For the fundamental unit vectors $\hat{i}.\hat{j}\times\hat{k}=1$, $\hat{i}.\hat{j}\times\hat{i}=0$.

## Vector Triple Product*

| | |
|---|---|
| **Vector Triple Product** | $\vec{A}\times\left(\vec{B}\times\vec{C}\right)=\left(\vec{A}.\vec{C}\right)\vec{B}-\left(\vec{A}.\vec{B}\right)\vec{C}$ |

* Only definition is included, further discussion is out of syllabus.



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ADDITIONAL TOPIC

# LOGARITHMIC AND EXPONENTIAL EQUATIONS

## Logarithm

If for a positive real number $(a \neq 1)$, $a^m = b$, then the index $m$ is called the logarithm of $b$ to the base $a$. We write it as

$$\log_a b = m$$

'log' being the abbreviation of the word '**logarithm**'. Thus,

$$a^m = b \Leftrightarrow \log_a b = m$$

where, $a^m = b$ is called the **exponential form** and $\log_a b = m$ is called the **logarithmic form**.

Log with base = 10 is called common log represented by 'log'
Log with base = '$e$' is called natural log represented by ' ln '

| Exponential form | Logarithmic form |
|---|---|
| $3^5 = 243$ | $\text{Log}_3 243 = 5$ |
| $2^4 = 16$ | $\text{Log}_2 16 = 4$ |
| $3^0 = 1$ | $\text{Log}_3 1 = 0$ |
| $8^{\frac{1}{3}} = 2$ | $\log_8 2 = \frac{1}{3}$ |

## Laws of Logarithm

| | |
|---|---|
| Product Formula | $\log_a (mn) = \log_a m + \log_a n$, where, $a$, $m$ are positive and $a \neq 1$ |
| Quotient Formula | $\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n$ where, $a$, $m$ are positive and $a \neq 1$ |
| Power Formula | $\log_a (m^n) = n \log_a m$, where, $a$, $m$ are positive and $a \neq 1$ |
| Base changing Formula | $\log_n m = \frac{\log_a m}{\log_a n}$. $So, \log_n m = \frac{\log m}{\log n}$ where $m$, $n$, $a$ are positive and $n \neq 1$, $a \neq 1$ |
| Reciprocal relation | $\log_b a \times \log_a b = 1$ Where $a$, $b$ are positive and not equal to 1. $\log_b a = \frac{1}{\log_a b}$ |
| One – to – one property | If $\log_a m = \log_a n$, *then* $m = n$ |



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## Key Facts

| | |
|---|---|
| **1.** | Logarithm of 1 to any base is equal to zero. i.e., $\log_a 1 = 0$ , where $a > 0, a \neq 0$ |
| **2.** | Logarithm of any number to the same base is 1. i.e., $\log_a a = 1$, where $a > 0, a \neq 1$ |
| **3.** | $a^{\log_a x} = x$ , where $a$ and $x$ are positive, $a \neq 1$ |

## Useful Results

| | |
|---|---|
| **1.** | $\log_{10} 10 = 1$, since $10^1 = 10$, |
| **2.** | $\log_{10} 100 = 2$, since $10^2 = 100$, |
| **3.** | $\log_{10} 10000 = 4$, since $10^4 = 10000$, |
| **4.** | $\log_{10} 0.01 = -2$, since $10^{-2} = 0.01$, |
| **5.** | $\log_{10} 0.001 = -3$, since $10^{-3} = 0.001$, |
| **6.** | $\log_{10} 1 = 0$, since $10^0 = 1$, |

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ADDITIONAL TOPIC

# QUADRATIC INEQUALITIES

## INTRODUCTION

Inequalities are expressed by the following four symbols:

$>$ (greater than) ; $<$ (less than) ;

$\geq$ (greater than or equal to);

$\leq$ (less than or equal to).

## Working procedure for solving a Quadratic Inequality

**(i)** To solve an inequality, we first convert the inequality into an equation. The equation thus formed is called the associated equation or the corresponding equation.

**(ii)** Solutions of the associated equation are called boundary numbers for the inequality.

**(iii)** If a rational expression occurs in an inequality, then the numbers where the denominator vanishes are not points in the domain of the rational expression. These numbers are called free boundary numbers.

**(iv)** Boundary numbers are part of the solution set if equality sign also occurs in the given inequality, but free boundary numbers are never part of the solution set since the given expression is not defined at these points.

**(v)** After finding the boundary numbers, locate them on the real line. The real line will be divided into a number of distinct regions.

**(vi)** To determine which of the regions belong to the solution set, we check the inequality by an arbitrary test point selected from each region.

**(vii)** The regions whose test points satisfy the inequality are in the solution set.

**(viii)** Union of all such region constitutes the solution set of the inequality.

## Tricks

| | |
|---|---|
| 1 | $(x-a)(x-b)<0 \Rightarrow x<a$ or $x>b$, for $a<b$ |
| 2 | $(x-a)(x-b)<0 \Rightarrow a<x<b$, for $a<b$ |
| 3 | $\lvert x\rvert<a \Rightarrow -a<x<a$ |
| 4 | $\lvert x\rvert>a \Rightarrow x<-a$ or $x>a$ |



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