Studybeesofficial

Where Minds Pollinate

Class 11

Mathematics

Chapter 7

Exercise 7.1

Notes

National Book Foundation

Follow Our Socials

studybeesofficial

studybeesofficialpak

studybeesofficialpak

Unit # 07

Mathematical Induction and Binomial Theorem

Exercise # 7.1

By method of mathematical induction prove the following when n is an integer.

1. $1+2+3+\ldots+n = \frac{n(n+1)}{2} \quad \forall\, n \geq 1$

Proof:-

case - 1: For $n = 1$, we have;

$$1 = \frac{1(1+1)}{2}$$

$$1 = 1$$

Inductive case: For $n = k$ i.e.

$$1+2+3+\ldots+k = \frac{k(k+1)}{2} \quad \text{—(i)}$$

Now, for $n = k+1$ the statement becomes

$$1+2+3+\ldots+k+(k+1) = \frac{(k+1)(k+1+1)}{2}$$

$$1+2+3+\ldots+(k+1) = \frac{(k+1)(k+2)}{2} \quad \text{—(ii)}$$

Now add $(k+1)$ on b/s in eq.(i)

$$\Rightarrow \quad 1+2+3+\dots+k+(k+1) = \frac{k(k+1)}{2} + (k+1)$$

$$= \frac{k(k+1)+2(k+1)}{2}$$

$$1+2+3+\dots+(k+1) = \frac{(k+1)(k+2)}{2} \quad \text{— (iii)}$$

From eq(ii) and (iii), the given sum is correct for all values of $n \geq 1$

2. $1^2+2^2+3^2+\dots+n^2 = \dfrac{n(n+1)(2n+1)}{6} \quad \forall\, n \geq 1$

Proof:-

Case - 1: For $n = 1$, we have;

$$1^2 = \frac{1(1+1)[2(1)+1)}{6}$$

$$1 = \frac{1(2)(3)}{6}$$

$$1 = 1$$

Inductive case: For $n=k$, i.e

$$1^2+2^2+3^2+\dots+k^2 = \frac{k(k+1)(2k+1)}{6} \quad \text{— (i)}$$

Now, for $n = k+1$. the statement becomes

i.e. $1^2+2^2+3^2+\dots+(k)^2+(k+1)^2 = \dfrac{(k+1)(k+1+1)[2(k+1)+1]}{6}$

$$1^2+2^2+3^2+\dots+(k+1)^2 = \frac{(k+1)(k+2)(2k+2+1)}{6}$$

$$1^2+2^2+3^2+\dots+(k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6} \quad -(ii)$$

Add $(k+1)^2$ on b/s in eq.(i)

$$\Rightarrow 1^2+2^2+3^2+\dots+k^2+(k+1)^2 = \frac{k(k+1)(2k+1)}{6}+(k+1)^2$$

$$= \frac{k(k+1)(2k+1)+6(k+1)^2}{6}$$

$$= \frac{(k+1)[k(2k+1)+6(k+1)]}{6}$$

$$1^2+2^2+3^2+\dots+(k+1)^2 = \frac{(k+1)(2k^2+k+6k+6)}{6}$$

$$= \frac{(k+1)(2k^2+7k+6)}{6}$$

$$1^2+2^2+3^2+\dots+(k+1)^2 = \frac{(k+1)[2k^2+4k+3k+6]}{6}$$

$$= \frac{(k+1)[2k(k+2)+3(k+2)]}{6}$$

$$1^2+2^2+3^2+\dots+(k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6} \quad -(iii)$$

From (ii) and (iii), the given sum is true.

3. $1^3+2^3+3^3+\ldots+n^3 = \dfrac{n^2(n+1)^2}{4} \quad \forall n \geq 1$

Proof:-

Case-1: For $n=1$, we have;

$$1^3 = \frac{(1)^2(1+1)^2}{4}$$

$$1 = \frac{1(4)}{4}$$

$$1 = 1$$

It is true.

Inductive step: For $n=k$, i.e.

$$1^3+2^3+3^3+\ldots+k^3 = \frac{k^2(k+1)^2}{4} \quad \text{—(i)}$$

Now, for $n=k+1$, the statement becomes,

$$1^3+2^3+3^3+\ldots+k^3+(k+1)^3 = \frac{(k+1)^2(k+1+1)^2}{4}$$

$$1^3+2^3+3^3+\ldots+(k+1)^3 = \frac{(k+1)^2(k+2)^2}{4} \quad \text{—(ii)}$$

Add, $(k+1)^3$ on b/s in eq.(i)

$$\Rightarrow 1^3+2^3+3^3+\ldots+k^3+(k+1)^3 = \frac{k^2(k+1)^2}{4}+(k+1)^3$$

$$= \frac{k^2(k+1)^2+4(k+1)^3}{4}$$

$$= \frac{(k+1)^2\left[k^2+4(k+1)\right]}{4}$$

$$1^3+2^3+3^3+\dots+(k+1)^3 = \frac{(k+1)^2(k^2+4k+4)}{4}$$

$$= \frac{(k+1)^2\left[(k)^2+2(2)(k)+(2)^2\right]}{4}$$

$$1^3+2^3+3^3+\dots+(k+1)^3 = \frac{(k+1)^2(k+2)^2}{4} \quad \text{—(iii)}$$

From (ii) and (iii), the given sum is true.

4. $$\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\dots+\frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1}, \forall n \geq 1$$

Proof:-

Case - 1: For $n = 1$, we have;

$$\frac{1}{1.4} = \frac{1}{3(1)+1} \Rightarrow \frac{1}{4} = \frac{1}{4}$$

It is true.

Inductive case: For $n=k$, i.e.

$$\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\dots+\frac{1}{(3k-2)(3k+1)} = \frac{k}{3k+1} \quad \text{—(i)}$$

Now for, $n = k+1$, the statement becomes,

$$\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\dots+\frac{1}{(3k-2)(3k+1)}+\frac{1}{[3(k+1)-2][3(k+1)+1]}$$

$$= \frac{k+1}{3(k+1)+1}$$

$$\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\dots+\frac{k1}{(3k+1)(3k+4)}=\frac{k+1}{3k+4}$$

Now add $\frac{1}{(3k+1)(3k+4)}$ on b/s in

eq.(i)

$$\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\dots+\frac{1}{(3k-2)(3k+1)}+\frac{1}{(3k+1)(3k+4)}$$

$$=\frac{k}{3k+1}+\frac{1}{(3k+1)(3k+4)}$$

$$=\frac{k(3k+4)+1}{(3k+1)(3k+4)}$$

$$\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\dots+\frac{1}{(3k+1)(3k+4)}=\frac{3k^2+4k+1}{(3k+1)(3k+4)}$$

$$=\frac{3k^2+3k+k+1}{(3k+1)(3k+4)}$$

$$=\frac{3k(k+1)+1(k+1)}{(3k+1)(3k+4)}$$

$$=\frac{\cancel{(3k+1)}(k+1)}{\cancel{(3k+1)}(3k+4)}$$

$$\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\dots+\frac{1}{(3k+1)(3k+4)}=\frac{k+1}{3k+4} \quad \text{— (iii)}$$

From (ii) and (iii), the sum is true.

5. $1^2 + 3^2 + 5^2 + \dots + (2n-1)^2 = \dfrac{n(4n^2-1)}{3} \quad \forall\, n \geq 1$

Proof:

Case-1: For $n = 1$, we have;

$$1^2 = \frac{1[4(1)^2 - 1]}{3}$$

$$1 = \frac{1(3)}{3} \Rightarrow 1 = 1$$

It is true.

Inductive case: For $n = k$, i.e.

$$1^2 + 3^2 + 5^2 + \dots + (2k-1)^2 = \frac{k(4k^2-1)}{3} \quad \text{---(i)}$$

Now for $n = k+1$, the statement becomes,

$$1^2 + 3^2 + 5^2 + \dots + (2k-1)^2 + [2(k+1)-1]^2 = \frac{(k+1)[4(k+1)^2 - 1]}{3}$$

$$1^2 + 3^2 + 5^2 + \dots + (2k+1)^2 = \frac{(k+1)(4k^2+8k+3)}{3} \quad \text{---(ii)}$$

Add $(2k+1)^2$ on b/s in eq.(i)

$$\Rightarrow 1^2 + 3^2 + 5^2 + \dots + (2k-1)^2 + (2k+1)^2 = \frac{k(4k^2-1)}{3} + (2k+1)^2$$

$$= \frac{k(4k^2-1) + 3(2k+1)^2}{3}$$

$$= \frac{k[(2k)^2 - (1)^2] + 3(2k+1)^2}{3}$$

$$= \frac{k(2k+1)(2k-1) + 3(2k+1)^2}{3}$$

$$1^2+3^2+5^2+\dots+(2k+1)^2 = \frac{(2k+1)[k(2k-1)+3(2k+1)]}{3}$$

$$= \frac{(2k+1)(2k^2-k+6k+3)}{3}$$

$$= \frac{(2k+1)(2k^2+5k+3)}{3}$$

$$= \frac{(2k+1)(2k^2+2k+3k+3)}{3}$$

$$= \frac{(2k+1)[2k(k+1)+3(k+1)]}{3}$$

$$= \frac{[(2k+1)(2k+3)](k+1)}{3}$$

$$1^2+3^2+5^2+\dots+(2k+1)^2 = \frac{(k+1)(4k^2+8k+3)}{3} \quad \text{-(iii)}$$

From (ii) and (iii), the given sum is true.

6. $4^3 + 4^4 + 4^5 + \dots + 4^n = \frac{4^3(4^{n-2}-1)}{3}, \quad \forall n \geq 3$

Proof:

Case-1: For $n = 3$, we have,

$$4^3 = \frac{4^3(4^{3-2}-1)}{3}$$

$$4^3 = \frac{4^3(3)}{3}$$

$4^3 = 4^3$, It is true

Inductive case: For $n = k$, i.e.

$$4^3 + 4^4 + 4^5 + \dots + 4^k = \frac{4^3(4^{k-2} - 1)}{3} \quad \text{—(i)}$$

Now, for $n = k+1$, the statement becomes,

$$4^3 + 4^4 + 4^5 + \dots + 4^k + 4^{k+1} = \frac{4^3(4^{k+1-2} - 1)}{3}$$

$$4^3 + 4^4 + 4^5 + \dots + 4^{k+1} = \frac{4^3(4^{k-1} - 1)}{3} \quad \text{—(ii)}$$

Add 4^{k+1} on b/s in eq. (i)

$$\Rightarrow 4^3 + 4^4 + 4^5 + \dots + 4^k + 4^{k+1} = \frac{4^3(4^{k-2} - 1)}{3} + 4^{k+1}$$

$$4^3 + 4^4 + 4^5 + \dots + 4^{k+1} = \frac{4^3(4^{k-2} - 1) + 3 \cdot 4^{k+1}}{3}$$

$$= \frac{4^3(4^{k-2} - 1 + 3 \cdot 4^{k+1-3})}{3}$$

$$= \frac{4^3(4^{k-2} - 1 + 3 \cdot 4^{k-2})}{3}$$

$$= \frac{4^3\left[4^{k-2}(3+1) - 1\right]}{3}$$

$$= \frac{4^3(4 \cdot 4^{k-2} - 1)}{3}$$

$$4^3 + 4^4 + 4^5 + \dots + 4^{k+1} = \frac{4^3(4^{k-1} - 1)}{3} \quad \text{—(iii)}$$

From (ii) and (iii), the given sum is true.

7. $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\dots+\frac{1}{n(n+1)}=\frac{n}{n+1}$, $\forall n\geq 1$

Proof:

Case-1: For $n=1$, we have,

$$\frac{1}{1.2}=\frac{1}{1(1+1)}$$

$$\frac{1}{2}=\frac{1}{2}\text{, It is true.}$$

Inductive case: For $n=k$, i.e.

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\dots+\frac{1}{k(k+1)}=\frac{k}{k+1} \quad \text{—(i)}$$

Now, for $n=k+1$, the statement becomes,

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\dots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+1+1)}=\frac{k+1}{k+1+1}$$

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\dots+\frac{1}{(k+1)(k+2)}=\frac{k+1}{k+2} \quad \text{—(ii)}$$

Add $\frac{1}{(k+1)(k+2)}$ on b/s in eq.(i)

$$\Rightarrow \frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\dots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$$

$$\frac{1}{1.2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\dots+\frac{1}{(k+1)(k+2)}=\frac{k(k+2)+1}{(k+1)(k+2)}$$

$$=\frac{k^2+2k+1}{(k+1)(k+2)}$$

$$=\frac{(k+1)^{\cancel{2}}}{\cancel{(k+1)}(k+2)}$$

$$\frac{1}{1.2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\dots+\frac{1}{(k+1)(k+2)}=\frac{k+1}{k+2} \quad \text{— (iii)}$$

From (ii) and (iii), the given sum is true for $\forall\, n \geq 1$.

10. $7+77+777+\dots+7_{n \text{ times}}=\frac{7}{81}(10^{n+1}-9n-10)$

$\forall\, n \geq 1$

Proof:

Case - 1 : For $n = 1$, we have,

$$7=\frac{7}{81}(10^{1+1}-9(1)-10)$$

$$7=\frac{7}{\cancel{81}}(\cancel{81})$$

$$7=7$$

It is true.

Inductive case: For $n=k$, i.e.

$$7+77+777+\dots+7_{k\text{ times}} = \frac{7}{81}(10^{k+1}-9k-10) \quad \text{--- (i)}$$

Now, for $n=k+1$, the statement becomes,

$$7+77+777+\dots+7_k+7_{(k+1)\text{times}} = \frac{7}{81}(10^{k+1+1}-9(k+1)-10)$$

$$7+77+777+\dots+7_{(k+1)\text{ times}} = \frac{7}{81}(10^{k+2}-9k-9-10)$$

$$7+77+777+\dots+7_{(k+1)\text{times}} = \frac{7}{81}(10^{k+2}-9k-19) \quad \text{--- (ii)}$$

Now add $7_{(k+1)\text{ times}}$ on b/s in eq. (i)

$$\Rightarrow 7+77+777+\dots+7_k+7_{(k+1)\text{ times}} = \frac{7}{81}(10^{k+1}-9k-10)+7_{(k+1)\text{ times}} \quad \text{--- (iii)}$$

Now consider $7_{(k+1)\text{ times}}$

So, $7_{(k+1)\text{times}} = 777777\dots7_{(k+1)}$

$77777\dots7_{(k+1)} = 7+70+700+7{,}000+70{,}000+\dots$

It is a geometric series

as $S_n = \frac{a(r^n-1)}{r-1}$

$a=7$, $n=k+1$, $r=\frac{70}{7}=10$

$\therefore S_{k+1} = \frac{7(10^{k+1}-1)}{10-9}$

$$\therefore \quad S_{k+1} = \frac{7(10^{k+1}-1)}{9}$$

eq.(iii) $\Rightarrow$ $$7+77+777+\dots+7_{(k+1)\text{ times}} = \frac{7}{81}\left(10^{k+1}-9k-10\right) + \frac{7(10^{k+1}-1)}{9}$$

$$= 7\left[\frac{10^{k+1}-9k-10}{(9)^2} + \frac{10^{k+1}-1}{9}\right]$$

$$= 7\left[\frac{10^{k+1}-9k-10+9(10^{k+1}-1)}{(9)^2}\right]$$

$$= 7\left[\frac{10^{k+1}-9k-10+9\cdot 10^{k+1}-9}{81}\right]$$

$$= \frac{7}{81}\left[10^{k+1}(9+1)-9k-19\right]$$

$$= \frac{7}{81}\left(10^{k+1}\cdot 10-9k-19\right)$$

$$7+77+777+\dots+7_{(k+1)\text{ times}} = \frac{7}{81}\left(10^{k+2}-9k-19\right) \quad \text{—(iv)}$$

From (ii) and (iv), the sum is true $\forall\, n \geq 1$

8. $$\frac{3}{1\cdot 2\cdot 2} + \frac{4}{2\cdot 3\cdot 2^2} + \frac{5}{3\cdot 4\cdot 2^3} + \dots + \frac{n+2}{n(n+1)2^n} = 1 - \frac{1}{(n+1)\cdot 2^n}$$

$\forall\, n \geq 1$

Proof:

Case - 1: For $n = 1$, we have;

$$\frac{3}{1.2.2} = 1 - \frac{1}{(1+1)2^1}$$

$$\frac{3}{4} = 1 - \frac{1}{2.2} \Rightarrow \frac{3}{4} = \frac{3}{4}$$

It is true.

Inductive case: For $n = k$, i.e.

$$\frac{3}{1.2.2} + \frac{4}{2.3.2^2} + \dots + \frac{k+2}{k(k+1)2^k} = 1 - \frac{1}{(k+1).2^k} \quad \text{---(i)}$$

Now, $n = k+1$, the statement becomes,

$$\frac{3}{1.2.2} + \frac{4}{2.3.2^2} + \dots + \frac{k+2}{k(k+1)2^k} + \frac{k+1+2}{(k+1)(k+1+1)2^{k+1}}$$

$$= 1 - \frac{1}{(k+1+1)2^{k+1}}$$

$$\frac{3}{1.2.2} + \frac{4}{2.3.2^2} + \dots + \frac{k+3}{(k+1)(k+2)2^{k+1}} = 1 - \frac{1}{(k+2).2^{k+1}} \quad \text{---(ii)}$$

Add $\frac{k+3}{(k+1)(k+2).2^{k+1}}$ on b/s in eq.(i)

$$\frac{3}{1.2.2} + \frac{4}{2.3.2^2} + \dots + \frac{k+2}{k(k+1)2^k} + \frac{k+3}{(k+1)(k+2)2^{k+1}} = 1 - \frac{1}{(k+1).2^k} + \frac{k+3}{(k+1)(k+2).2^{k+1}}$$

$$= 1 - \frac{1}{(k+1)2^k} + \frac{k+3}{(k+1)(k+2)2^k.2^1}$$

$$= 1 - \frac{[2(k+2)] + k+3}{(k+1)(k+2)2^k.2^1}$$

$$= 1 - \frac{[2k+4] + k + 3}{(k+1)(k+2)2^{k+1}}$$

$$= 1 + \frac{\cancel{3}k - 1}{(k+1)(k+2)2^{k+1}}$$

$$= 1 - \frac{\cancel{(k+1)}}{\cancel{(k+1)}(k+2)2^{k+1}}$$

$$\frac{3}{1.2.2} + \frac{4}{2.3.2^2} + \dots + \frac{k+3}{(k+1)(k+2)2^{k+1}} = 1 - \frac{1}{(k+2)2^{k+1}} \quad \text{—(iii)}$$

From (ii) and (iii), the sum is true.

9. $$\frac{5}{1.2.3} + \frac{6}{2.3.4} + \frac{7}{3.4.5} + \dots + \frac{n+4}{n(n+1)(n+2)} = \frac{n(3n+7)}{2(n+1)(n+2)}$$

Proof: $\forall\ n \geq 1$

Case - 1: For $n = 1$, we have,

$$\frac{5}{1.2.3} = \frac{1(3(1)+7)}{2(1+1)(1+2)} \Rightarrow \frac{5}{6} = \frac{5}{6}$$

Result holds.

Inductive case: For $n=k$, i.e.

$$\frac{5}{1\cdot2\cdot3}+\frac{6}{2\cdot3\cdot4}+\frac{7}{3\cdot4\cdot5}+\cdots+\frac{k+4}{k(k+1)(k+2)}=\frac{k(3k+7)}{2(k+1)(k+2)}$$

Now for $n=k+1$, the statement becomes,

$$\frac{5}{1\cdot2\cdot3}+\frac{6}{2\cdot3\cdot4}+\frac{7}{3\cdot4\cdot5}+\cdots+\frac{k+4}{k(k+1)(k+2)}+\frac{k+1+4}{(k+1)(k+1+1)(k+1+2)}$$

$$=\frac{(k+1)(3(k+1)+7)}{2(k+1+1)(k+1+2)}$$

$$\frac{5}{1\cdot2\cdot3}+\frac{6}{2\cdot3\cdot4}+\frac{7}{3\cdot4\cdot5}+\cdots+\frac{k+5}{(k+1)(k+2)(k+3)}=\frac{(k+1)(3k+10)}{2(k+2)(k+3)} \quad -(ii)$$

Now add $\frac{k+5}{(k+1)(k+2)(k+3)}$ b/s in eq.(i)

$$\frac{5}{1\cdot2\cdot3}+\frac{6}{2\cdot3\cdot4}+\cdots+\frac{k+4}{k(k+1)(k+2)}+\frac{k+5}{(k+1)(k+2)(k+3)}$$

$$=\frac{\cancel{k(3k+7)}}{\cancel{2(k+1)(k+2)}}+\frac{\cancel{k+5}}{\cancel{(k+1)(k+2)(k+3)}}$$

$$=\frac{k(3k+7)(k+3)+2(k+5)}{2(k+1)(k+2)(k+3)}$$

$$=\frac{k(3k^2+9k+7k+21)+2k+10}{2(k+1)(k+2)(k+3)}$$

$$= \frac{3k^3+16k^2+21k+2k+10}{2(k+1)(k+2)(k+3)}$$

$$= \frac{3k^3+16k^2+23k+10}{2(k+1)(k+2)(k+3)}$$

$\because$ $16k^2 = 13k^2+3k^2$, $23k = 13k+10k$

$\therefore$
$$= \frac{3k^3+3k^2+13k^2+13k+10k+10}{2(k+1)(k+2)(k+3)}$$

$$= \frac{3k^2(k+1)+10(k+1)+13k(k+1)}{2(k+1)(k+2)(k+3)}$$

$$= \frac{\cancel{(k+1)}(3k^2+13k+10)}{2\cancel{(k+1)}(k+2)(k+3)}$$

$$= \frac{3k^2+3k+10k+10}{2(k+2)(k+3)}$$

$$= \frac{3k(k+1)+10(k+1)}{2(k+2)(k+3)}$$

$$\frac{5}{1.2.3}+\frac{6}{2.3.4}+\dots+\frac{k+5}{(k+1)(k+2)(k+3)} = \frac{(k+1)(3k+10)}{2(k+2)(k+3)} \quad \text{—(iii)}$$

From two (ii) and (iii), the results hold.

11. $1^3+3^3+5^3+\dots+(2n+1)^3 = (n+1)^2(2n^2+4n+1) \quad \forall\, n \geq 0$

Proof:

Case - 1: For $n=0$, we have

$$1^3 = (0+1)^2(2(0)^2+4(0)+1)$$
$$1 = 1 \quad \text{result holds}$$

Inductive case: For $n=k$, i.e.

$$1^3+3^3+5^3+\dots+(2k+1)^3 = (k+1)^2(2k^2+4k+1) \quad \text{—(i)}$$

Now, for $n=k+1$, the statement becomes,

$$1^3+3^3+5^3+\dots+(2k+1)^3+[2(k+1)+1]^3 = (k+1+1)^2\{2(k+1)^2+4(k+1)+1\}$$

$$1^3+3^3+5^3+\dots+(2k+3)^3 = (k+2)^2[2(k+1)^2+4(k+1)+1] \quad \text{—(ii)}$$

Add $(2k+3)^3$ on b/s in eq.(i)

$$\Rightarrow 1^3+3^3+5^3+\dots+(2k+1)^3+(2k+3)^3 = (k+1)^2(2k^2+4k+1)+(2k+3)^3$$

$$= (k^2+2k+1)(2k^2+4k+1)+(2k)^3+(3)^3 + 3(2k)(3)(2k+3)$$

$$= 2k^4+4k^3+k^2+4k^3+8k^2+2k+2k^2+4k+1 + 8k^3+27+36k^2+54k$$

$$= 2k^4+16k^3+47k^2+60k+28 \quad \text{—(iii)}$$

on solving eq.(ii) we also get eq.(iii) so, the results hold for the given sum.

12. $1.2^0 + 2.2^1 + 3.2^2 + \dots + n.2^{n-1} = (n-1).2^n + 1 \quad \forall n \geq 1$

Proof:

Case-1: For $n = 1$, we have,

$$1.2^0 = (1-1).2^1 + 1$$
$$1(1) = (0)2 + 1$$
$$1 = 1 \text{ result holds}$$

Inductive case: For $n = k$, i.e.

$$1.2^0 + 2.2^1 + 3.2^2 + \dots + k.2^{k-1} = (k-1)2^k + 1 \quad \text{— (i)}$$

Now, for $n = k+1$, the statement becomes,

$$1.2^0 + 2.2^1 + 3.2^2 + \dots + k.2^{k-1} + (k+1)2^{k+1-1} = (k+1-1)2^{k+1} + 1$$

$$1.2^0 + 2.2^1 + 3.2^2 + \dots + (k+1).2^k = (k).2^{k+1} + 1 \quad \text{— (ii)}$$

Add $(k+1).2^k$ on b/s in (i)

$$1.2^0 + 2.2^1 + 3.2^2 + \dots + k.2^{k-1} + (k+1).2^k = (k-1)2^k + 1 + (k+1).2^k$$

$$= (k-1)2^k.2^0 + (k+1).2^k + 1$$

$$= 2^k[2^0(k-1) + (k+1)] + 1$$

$$= 2^k(2k - \cancel{1} + k + \cancel{1}] + 1$$

$$= \cancel{2^k(2k)+1}$$

$1.2^0 + 2.2^1 + \ldots + (k+1)2^k = 2^{k+1}(k) + 1$ — (iii)

From (ii) and (iii), the results are true for

$\forall\, n \geq 1$

13. $1.1! + 2.2! + 3.3! + \ldots + n.n! = (n+1)! - 1 \quad \forall\, n \geq 1$

Proof:

Case - 1: For $n = 1$, we have,

$$1.1! = (1+1)! - 1$$
$$1(1) = 2! - 1$$
$$1 = 2 - 1$$
$$1 = 1 \text{ result holds.}$$

Inductive case: For $n = k$, i.e.

$1.1! + 2.2! + 3.3! + \ldots + k.k! = (k+1)! - 1$

Now, for $n = k+1$, the statement becomes.

$1.1! + 2.2! + 3.3! + \ldots + k.k! + (k+1)(k+1)! = (k+1+1)! - 1$

$1.1! + 2.2! + 3.3! + \ldots + (k+1)(k+1)! = (k+2)! - 1$ — (ii)

Add $(k+1)(k+1)!$ on b/s in eq. (i)

$\Rightarrow 1.1! + 2.2! + \ldots + k.k! + (k+1)(k+1)! = (k+1)! - 1 + (k+1)(k+1)!$

$$= (k+1)!\left[1+(k+1)\right]-1$$

$$= (k+1)!\,(k+1+1)-1$$

$$= (k+1)!\,(k+2)-1$$

$$= (k+2)(k+1)!-1$$

$$1.1!+2.2!+3.3!+\dots+(k+1)(k+1)! = (k+2)!-1 \quad \text{—(iv)}$$

From (ii) and (iii), the result holds $\forall\, n \geq 1$.

14. $$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{n^2}\right) = \frac{n+1}{2n}$$

$\forall\, n \geq 2$

Proof:

Case – 1: For $n=2$, we have,

$$\left(1-\frac{1}{2^2}\right) = \frac{2+1}{2(2)}$$

$$\frac{4-1}{4} = \frac{3}{2}$$

$$\frac{3}{2} = \frac{3}{2} \quad \text{result holds}$$

Inductive case: For $n=k$, i.e.

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{k^2}\right) = \frac{k+1}{2k} \quad \text{—(i)}$$

Now for $n=k+1$, the statement becomes

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{(k+1)^2}\right) = \frac{k+1+1}{2(k+1)}$$

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{(k+1)^2}\right) = \frac{k+2}{2k+2}$$

Now multiply $\left(1-\frac{1}{(k+1)^2}\right)$ on b/s in (i)

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{k^2}\right)\left(1-\frac{1}{(k+1)^2}\right) = \frac{k+1}{2k}\left(1-\frac{1}{(k+1)^2}\right)$$

$$= \frac{\cancel{k+1}}{2k}\left[\frac{(k+1)^2-1}{(k+1)^{\cancel{2}}}\right]$$

$$= \frac{(k+1)^2-1}{2k(k+1)}$$

$$= \frac{k^2+2k+\cancel{1}-\cancel{1}}{2k(k+1)}$$

$$= \frac{k^2+2k}{2k(k+1)}$$

$$= \frac{\cancel{k}(k+2)}{2\cancel{k}(k+1)}$$

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{(k+1)^2}\right) = \frac{k+2}{2k+2} \quad \text{--- (iv)}$$

From (ii) and (iv), the results are true.

15. $\left(\frac{1}{1}\cdot\frac{1}{2}\right)\left(\frac{1}{3}\cdot\frac{1}{4}\right)\left(\frac{1}{5}\cdot\frac{1}{6}\right)\cdots\left(\frac{1}{2n+1}\cdot\frac{1}{2n+2}\right)=\frac{1}{(2n+2)!}$

$\forall n \geq 0$

Proof:

Case-1: For $n=0$, we have

$$\left(\frac{1}{1}\cdot\frac{1}{2}\right)=\frac{1}{(2(0)+2)!}$$

$$\frac{1}{2}=\frac{1}{2!} \Rightarrow \frac{1}{2}=\frac{1}{2}$$ ~~result holds~~

Inductive case:- For $n=k$, i.e.

$$\left(\frac{1}{1}\cdot\frac{1}{2}\right)\left(\frac{1}{3}\cdot\frac{1}{4}\right)\left(\frac{1}{5}\cdot\frac{1}{6}\right)\cdots\left(\frac{1}{2k+1}\cdot\frac{1}{2k+2}\right)=\frac{1}{(2k+2)!} \quad -(i)$$

Now for $n=k+1$, the statement becomes,

$$\left(\frac{1}{1}\cdot\frac{1}{2}\right)\left(\frac{1}{3}\cdot\frac{1}{4}\right)\cdots\left(\frac{1}{2(k+1)+1}\cdot\frac{1}{2(k+1)+2}\right)=\frac{1}{[2(k+1)+2]!}$$

$$\left(\frac{1}{1}\cdot\frac{1}{2}\right)\left(\frac{1}{3}\cdot\frac{1}{4}\right)\cdots\left(\frac{1}{2k+3}\cdot\frac{1}{2k+4}\right)=\frac{1}{(2k+4)!} \quad -(ii)$$

Multiply $\frac{1}{(2k+3)(2k+4)}$ on b/s in eq.(i)

$$\Rightarrow \left(\frac{1}{1}\cdot\frac{1}{2}\right)\left(\frac{1}{3}\cdot\frac{1}{4}\right)\cdots\left(\frac{1}{2k+1}\cdot\frac{1}{2k+2}\right)\left(\frac{1}{2k+3}\cdot\frac{1}{2k+4}\right)$$

$$=\frac{1}{(2k+2)!}\left[\frac{1}{(2k+3)(2k+4)}\right]$$

$$= \frac{1}{(2k+4)(2k+3)(2k+2)!}$$

$$\left(\frac{1}{1}\cdot\frac{1}{2}\right)\left(\frac{1}{3}\cdot\frac{1}{4}\right)\left(\frac{1}{5}\cdot\frac{1}{6}\right)\cdots\left(\frac{1}{2k+3}\cdot\frac{1}{2k+4}\right) = \frac{1}{(2k+4)!} \quad \text{— (iii)}$$

From (ii) and (iii) the result holds $\forall\, n \geq 0$

16. $1-2+2^2-2^3+\ldots+(-1)^n 2^n = \dfrac{1-(-2)^{n+1}}{3} \quad \forall n \geq 0$

Proof:

Case – 1: For $n = 1$, we have,

$$1 = \frac{1-(-2)^{(0)+1}}{3}$$

$$1 = \frac{1+2}{3}$$

$1 = 1$ result holds.

Inductive case:

As, $1-2+2^2-2^3+\ldots+(-1)^n 2^n = \dfrac{1-(-2)^{n+1}}{3}$

OR $2^0+(-2)^1+2^2+(-2)^3+\ldots+(-1)^n 2^n = \dfrac{1-(-2)^{n+1}}{3}$

For $n = k$, i.e.

$$2^0+(-2)^1+(2)^2+(-2)^3+\ldots+(-1)^k(2)^k = \frac{1-(-2)^{k+1}}{3} \quad \text{— (i)}$$

Now for $n=k+1$, the statement becomes,

$$2^0+(-2)^1+2^2+(-2)^3+\dots+(-1)^{k+1}(2)^{k+1} = \frac{1-(-2)^{k+2}}{3} \quad -(ii)$$

Now add $(-1)^{k+1}(2)^{k+1}$ on b/s in (i)

$$\Rightarrow 2^0+(-2)^1+2^2+(-2)^3+\dots+(-1)^k(2)^k+(-1)^{k+1}(2)^{k+1} = \frac{1-(-2)^{k+1}}{3}+(-1)^{k+1}(2)^{k+1}$$

$$= \frac{1-(-2)^{k+1}+3(-1)^{k+1}2^{k+1}}{3}$$

$$= \frac{1-(-2)^{k+1}+3(-1)^{k+1}2^{k+1}}{3}$$

$$= \frac{1-(-2)^{k+1}+3(-1\cdot 2)^{k+1}}{3}$$

$$= \frac{1-(-2)^{k+1}+3(-2)^{k+1}}{3}$$

$$= \frac{-(-2)^{k+1}(1-3)+1}{3}$$

$$= \frac{-(-2)^{k+1}(-2)+1}{3}$$

$$= \frac{1-(-2)^{k+1+1}}{3}$$

$$2^0+(-2)^1+2^2+(-2)^3+\dots+(-1)^{k+1}(2)^{k+1} = \frac{1-(-2)^{k+2}}{3} \quad -(iii)$$

From (ii) and (iii), the results are true $\forall\, n \geq 0$

Prove the following by mathematical induction.

20. n^3+2n is divisible by 3 $\forall\, n \geq 1$

Proof:

Case - 1:- For $n=1$, we have,

$= (1)^3+2(1)$
$= 1+2$
$= 3$ is divisible by 3.

Inductive case:- suppose for $n=k$ is also true

$\therefore \quad = k^3+2k$ is divisible by 3.

Now for $n=k+1$, i.e.

$= (k+1)^3+2(k+1)$
$= (k+1)[(k+1)^2+2]$
$= (k+1)(k^2+2k+1+2)$
$= (k+1)(k^2+2k+3)$
$= k^3+2k^2+3k+k^2+2k+3$
$= k^3+3k^2+5k+3$
$= k^3+2k+3k+3k^2+3$
$= (k^3+2k)+3(k+k^2+1)$
$= (k^3+2k)+3(k^2+k+1)$

In above expression 1^{st} term is divisible by 3 accroding to our supposition and in 2^{nd} is multiple of 3 – so, it is divisible by 3 and result holds fon $n=k+1$ also.

21. 6 is a factor of $n(n^2+5)$ $\forall n \geq 1$

Proof:

6 is factor of $n(n^2+5)$ means it is also divisible by $n(n^2+5)$

Case- 1: For $n = 1$, we have,

$= 1(1^2+5)$

$= 1(6)$

$= 6$ is divisible by 6.

Inductive case: For $n=k$, suppose it is also true

$\therefore = k(k^2+5)$ is divisible by 6.

Now for $n = k+1$, i.e

$= (k+1)[(k+1)^2+5]$

$= (k+1)(k^2+2k+1+5)$

$= (k+1)(k^2+2k+6)$

$= (k^3+2k^2+6k+k^2+2k+6$

$= k^3+3k^2+8k+6$

$= k^3+3k^2+5k+3k+6$

$= k^3+5k+3k^2+3k+6$

$= k(k^2+5)+3(k^2+k+2)$

$= k(k^2+5)+3(k^2+k+2)\frac{2}{2}$

$= k(k^2+5)+6\left(\frac{k^2+k+2}{2}\right)$

Hence, it is also true.

22. $\dfrac{n(3n^4+5n^2+7)}{15}$ is a rational no.

Proof:-

Case - 1: For $n=1$, we have,

$$= \frac{1[3(1)^4+5(1)^2+7]}{15}$$

$= 1$ is a rational number.

Inductive case: Suppose the no. is also rational for $n=k$.

$$\therefore \quad = \frac{k(3k^4+5k^2+7)}{15}$$

Now for $n=k+1$, i.e.

$$= \frac{(k+1)[3(k+1)^4+5(k+1)^2+7]}{15}$$

$$= \frac{(k+1)[3(k^2+2k+1)(k^2+2k+1)+5(k^2+2k+1)+7]}{15}$$

$$= \frac{(k+1)[3(k^4+2k^3+k^2+2k^3+4k^2+2k+k^2+2k+1) + 5k^2+10k+7+5]}{15}$$

$$= \frac{(k+1)}{15}\left[3k^4+12k^3+18k^2+12k+3+5k^2+10k+5+7\right]$$

$$= \frac{k+1}{15}\left[(3k^4+5k^2+7)+(12k^3+18k^2+22k+8)\right]$$

$$= \frac{k}{15}\left[3k^4+5k^2+7\right] + \frac{k}{15}\left[12k^3+18k^2+22k+8\right] + \frac{1}{15}\left[3k^4+5k^2+7+12k^3+18k^2+22k+8\right]$$

$$= \frac{k}{15}(3k^4+5k^2+7) + \frac{1}{15}\left[12k^4+18k^3+22k^2+8k+3k^4+5k^2+7+12k^3+18k^2+22k+8\right]$$

$$= \frac{k}{15}(3k^4+5k^2+7) + \frac{1}{15}\left[15k^4+30k^3+45k^2+30k+15\right]$$

$$= \frac{k(3k^4+5k^2+7)}{15} + \frac{\cancel{15}}{\cancel{15}}\left[k^4+2k^3+3k^2+2k+1\right]$$

$$= \frac{k(3k^4+5k^2+7)}{15} + k^4+2k^3+3k^2+2k+1$$

In above expression first term is rational no. acc. to our previous supposition and second term is an integer which is also rational. So, the sum of two rational numbers is also rational.

Therefore, given expression is a rational no.

23. $4^n + 15n - 1$ is divisible by 9 $\forall n \geq 1$

Proof:

Case-1: For $n=1$, we have,

$= 4^1 + 15(1) - 1$
$= 4 + 15 - 1$
$= 18$ is divisible by 9.

Inductive case: Suppose given expression is also divisible by 9 for $n=k$.

$\therefore = 4^k + 15k - 1$

Now for, $n = k+1$

$\therefore = 4^{(k+1)} + 15(k+1) - 1$
$= 4^{k+1} + 15k + 15 - 1$
$= 4^k \cdot 4 + 15k + 15 - 1$

Add and subtract $(45k - 3)$.

$\therefore = 4^k \cdot 4 + 15k + 15 - 1 + 45k - 3 - 45k + 3$
$= (4^k \cdot 4 + 15k + 45k - 1 - 3) + 15 - 45k + 3$
$= 4(4^k + 15k - 1) - 45k + 18$
$= 4(4^k + 15k - 1) + 9(2 - 5k)$

Hence, it is divisible by 9 also.

24. $7^n - 2^n$ is divisible by 5 $\forall n \geq 0$

Proof:

Case - 1: For $n = 0$, we have,

$= 7^0 - 2^0$

$= 0$ is divisible by 5.

Inductive case: Suppose given expression is also divisible by 5 for $n = k$.

$\therefore = 7^k - 2^k$ is divisible by 5.

Now for $n = k+1$ i.e.

$\therefore = 7^{k+1} - 2^{k+1}$

$= 7^k . 7^1 - 2^k . 2^1$

$= 7^k(7) - 2^k . 2$

$= 7^k(5+2) - 2^k . 2$

$= 5.7^k + 2.7^k - 2^k . 2$

$= 5.7^k + 2(7^k - 2^k)$

$= 2(7^k - 2^k) + 5(7^k)$

Hence, it is also divisible by 5 also.

17. $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\dots+\binom{n}{n} = 2^n \quad \forall n \geq 0$

Proof:

Basis step: For $n=0$, we have,

$$\binom{0}{0} = 2^0$$

$$^0C_0 = 1$$

OR $\dfrac{0!}{(0-0)!\,0!} = 1$

$1 = 1$ result holds.

Inductive step: Put $n=k$

i.e. $\binom{k}{0}+\binom{k}{1}+\binom{k}{2}+\dots+\binom{k}{k} = 2^k \quad -(i)$

Now put $n = k+1$, we have,

$$\binom{k+1}{0}+\binom{k+1}{1}+\binom{k+1}{2}+\dots+\binom{k+1}{k+1} = 2^{k+1}$$

Multiply 2 on b/s in eq.(i)

$\Rightarrow \quad 2\binom{k}{0}+2\binom{k}{1}+2\binom{k}{2}+\dots+2\binom{k}{k} = 2\cdot 2^k$

$$\binom{k}{0}+\left[\binom{k}{0}+\binom{k}{1}\right]+\left[\binom{k}{1}+\binom{k}{2}\right]+\dots+\binom{k}{k}\binom{k}{k}$$

$$= 2^{k+1}$$

$\because \binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$

$\therefore \binom{k}{0} + \binom{k+1}{1} + \binom{k+1}{2} + \dots + \binom{k+1}{k} + \binom{k}{k} = 2^{k+1}$

$\because \binom{n}{0} = \binom{n+1}{0} = \binom{n+1}{n+1}$

$\therefore \binom{k+1}{0} + \binom{k+1}{1} + \binom{k+1}{2} + \dots + \binom{k+1}{k+1} = 2^{k+1}$

Hence proved.

18. $\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \dots + n\binom{n}{n} = n.2^{n-1} \quad \forall n \geq 1$

Proof:

Basis step: For $n = 1$, we have

$\binom{1}{1} = (1)2^{1-1}$

$1 = 1(1)$

$1 = 1$ result holds

Inductive step: For $n = k$, i.e.

$\binom{k}{1} + 2\binom{k}{2} + 3\binom{k}{3} + \dots + k\binom{k}{k} = k.2^{k-1} \quad -(1)$

Now, for $n = k+1$, we have;

$$\binom{k+1}{1} + 2\binom{k+1}{2} + 3\binom{k+1}{3} + \dots + (k+1)\binom{k+1}{k+1} = (k+1).2^{k+1-1}$$

$$\binom{k+1}{1} + 2\binom{k+1}{2} + 3\binom{k+1}{3} + \dots + (k+1)\binom{k+1}{k+1} = 2^k(k+1)$$

Multiply 2 on b/s and add 2^k in eq.(i)

$$\Rightarrow 2\left[\binom{k}{1} + 2\binom{k}{2} + 3\binom{k}{3} + \dots + k\binom{k}{k}\right] + 2^k = 2^k + 2(k.2^{k-1})$$

$$2^k + 2\binom{k}{1} + 4\binom{k}{2} + 6\binom{k}{3} + \dots + 2k\binom{k}{k} = (k+1)2^k$$

As,

$$2^k = \binom{k}{0} + \binom{k}{1} + \dots + \binom{k}{k}$$

$$\left[\binom{k}{0} + \binom{k}{1} + \binom{k}{2} + \dots + \binom{k}{k}\right] + 2\binom{k}{1} + 4\binom{k}{2} + 6\binom{k}{3} + \dots + 2k\binom{k}{k} = (k+1)2^k$$

$$\binom{k}{0} + \binom{k}{1} + \binom{k}{2} + \dots + \binom{k}{k} + 2\binom{k}{1} + 2\binom{k}{2} + 2\binom{k}{2}$$

$$+ 3\binom{k}{2} + 3\binom{k}{3} + \dots + k\binom{k}{k} + k\binom{k}{k} = (k+1)2^k$$

$$\left[\binom{k}{0} + \binom{k}{1}\right] + 2\left[\binom{k}{1} + \binom{k}{2}\right] + 3\left[\binom{k}{2} + \binom{k}{3}\right]$$

$$+ \dots + k\binom{k}{k} + (1+k)\binom{k}{k} = (k+1)2^k$$

$$\binom{k+1}{1} + 2\binom{k+1}{2} + 3\binom{k+1}{3} + \dots + (k+1)\binom{k+1}{k+1} = (k+1)2^{k}$$

Hence proved.

19. $$\binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \dots + \frac{1}{n+1}\binom{n}{n} = \frac{2^{n+1}-1}{n+1}$$

$\forall\ n \geq 0$

Proof:

Basis step: For $n=0$, we have

$$\binom{0}{0} = \frac{2^{0+1}-1}{0+1}$$

$$\frac{0!}{(0-0)!\,0!} = \frac{2-1}{1}$$

$1 = 1$ results hold.

Inductive step: For $n=k$, i.e

$$\binom{k}{0} + \frac{1}{2}\binom{k}{1} + \frac{1}{3}\binom{k}{2} + \dots + \frac{1}{k+1}\binom{k}{k} = \frac{2^{k+1}-1}{k+1}$$

Now for $n=k+1$, we have;

$$\binom{k+1}{0} + \frac{1}{2}\binom{k+1}{1} + \frac{1}{3}\binom{k+1}{2} + \dots + \frac{1}{k+1+1}\binom{k+1}{k+1} = \frac{2^{k+1+1}-1}{k+1+1}$$

$$\binom{k+1}{0} + \frac{1}{2}\binom{k+1}{1} + \frac{1}{3}\binom{k+1}{2} + \dots + \frac{1}{k+2}\binom{k+1}{k+1} = \frac{2^{k+2}-1}{k+2}$$

use,

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\binom{n}{3}+\dots+\binom{n}{n}=2^n$$

Put $n=k+2$

$$\therefore \binom{k+2}{0}+\binom{k+2}{1}+\binom{k+2}{2}+\dots+\binom{k+2}{k+2}=2^{k+2}$$

$$\because \binom{k+2}{0}=1$$

$$\therefore \binom{k+2}{1}+\binom{k+2}{2}+\binom{k+2}{3}+\dots+\binom{k+2}{k+2}=2^{k+2}-1$$

$$\frac{(k+2)!}{(k+2-1)!\,1!}+\frac{(k+2)!}{(k+2-2)!\,2!}+\dots+\frac{(k+2)!}{(k+2)!\,0!}=2^{k+2}-1$$

$$\frac{(k+2)(k+1)!}{(k+1)!\,1!}+\frac{(k+2)(k+1)!}{(k)!\,2!}+\dots+\frac{(k+2)(k+1)!}{(k+2)!}=2^{k+2}-1$$

$$(k+2)\left[\frac{\cancel{(k+1)!}}{\cancel{(k+1)!}}+\frac{(k+1)!}{2k!}+\dots+\frac{(k+1)!}{(k+2)!}\right]=2^{k+2}-1$$

$$1+\frac{(k+1)!}{2k!}+\dots+\frac{(k+1)!}{(k+2)!}=\frac{2^{k+2}-1}{k+2}$$

$$\because \binom{n+1}{0}=1$$

$$\therefore \binom{k+1}{0}+\frac{1}{2}\binom{k+1}{1}+\frac{1}{3}\binom{k+1}{2}+\dots+\frac{1}{k+2}\binom{k+1}{k+1}=\frac{2^{k+2}-1}{k+2}$$

Hence proved.

Prove the following inequalities by mathematical induction.

25) $2^n < (n+1)! \quad \forall\, n \geq 2$

Proof:

Basis step: For $n=2$, we have,

$$2^2 < (2+1)!$$
$$4 < 3!$$
$$4 < 6 \quad \text{results hold.}$$

Inductive step: For $n=k$, i.e.

$$2^k < (k+1)! \quad \text{—(i)}$$

Suppose it is true

Now for $n = k+1$

$$\Rightarrow \quad 2^{k+1} < (k+1+1)!$$
$$\cancel{2^{k+1} < (k+2)!}$$

~~If $2 < (k+2)$~~

Multiply 2 with left side of (i) and $(k+2)$ with right side then inequality will be same

$$\Rightarrow \quad 2.2^k < (k+2)(k+1)!$$
$$2^{k+1} < (k+2)!$$

Hence proved.

32) $n! > n^2 \quad \forall n \geq 4$

Proof:

Basis step: For $n = 4$, we have,

$$4! > 4^2$$

$$24 > 16$$ result holds

Inductive step: For $n = k$, i.e.

$$k! > k^2 \quad -(i)$$

suppose results hold.

Now put $n = k+1$, we have;

$$(k+1)! > (k+1)^2$$

As $(k+1) > 1-2k \rightarrow$ true

Now multiply $(k+1)$ on left side and $(1-2k)$ add on right side so inequality (i) will be same.

$$\therefore \quad (k+1)k! > k^2 + 1 - 2k$$

$$(k+1)! > (k+1)^2$$

Hence proved.

31) $n^3 > 2n+1 \quad \forall n \geq 2$

Proof:

Basis step: For $n = 2$, we have;

$$(2)^3 > 2(2)+1$$

~~$8 > 5$ result holds~~

Inductive step: For $n=k$, i.e.

$\Rightarrow \quad k^3 > 2k+1 \quad -(i)$

~~suppose it is true~~

Now for, $n=k+1$, we have;

$\therefore \quad (k+1)^3 > 2(k+1)+1$

$$\cancel{(k+1)^3} > 2k+3$$

Add, $1+3k^2+3k$ on l.h.s in (i)

So, add $1+3k^2+3k$ on left side and

~~1 on right side~~

$(k+1)^3 > 3k^2+5k$

$\Rightarrow \quad k^3+1+3k^2+3k > 2k+1+1$

$$(k+1)^3 > 2k+3$$

Hence proved.

30. $1+3n \leq 4^n \quad \forall\ n \geq 0$

Proof:

Basis step: For $n=0$, we have

$$1+3(0) \leq 4^0$$
$$1+0 \leq 1$$
$$1 \leq 1 \quad \text{results hold}$$

Inductive step: For $n=k$, we have;

$$\Rightarrow \quad 1+3k \leq 4^k \quad -(i)$$

Suppose it is true

Now for $n=k+1$, i.e.

$$\therefore \quad 1+3(k+1) \leq 4^{k+1}$$
$$1+3k+3 \leq 4^{k+1}$$
$$3k+4 \leq 4^{k+1}$$

Multiply 4 on b/s in (i)

$$4(1+3k) \leq 4.4^k$$
$$4+12k \leq 4^{k+1}$$
$$4+3k+9k \leq 4^{k+1}$$

neglecting 9k from inequality will not effect it

$$\Rightarrow \quad 3k+4 \leq 4^{k+1}$$

Hence proved.

$\sqrt[n]{n} < 2 - \frac{1}{n} \quad \forall\, n \geq 2$

Q9.

Proof:

Basis step: For $n = 2$, we have;

$$\Rightarrow \sqrt[2]{2} < 2 - \frac{1}{2}$$

$1.41 < 1.5$ result holds

Inductive step: For $n = k$, we have;

$$\therefore \sqrt[k]{k} < 2 - \frac{1}{k} \quad -(i)$$

for $n = k+1$, we have;

$$\therefore \sqrt[k+1]{k+1} < 2 - \frac{1}{k+1}$$

As, $(k+1)^{1/k+1} < (k)^{1/k}$, $\quad 2 - \frac{1}{k} < 2 - \frac{1}{k+1}$

OR $\quad (k+1)^{1/k+1} < k^{1/k} < 2 - \frac{1}{k} < 2 - \frac{1}{k+1}$

$$\Rightarrow (k+1)^{1/k+1} < 2 - \frac{1}{k+1}$$

$$\sqrt[k+1]{k+1} < 2 - \frac{1}{k+1}$$

Hence proved.

28. $\binom{2n}{n} < 2^{2n-2} \quad \forall n \geq 5$

Proof:-

Basis setp: For $n = 5$

$$\binom{2(5)}{5} \leq 2^{2(5)-2}$$

$$252 \leq 256$$

Inductive step: For $n=k$, i.e.

$$\therefore \binom{2k}{k} < 2^{2k-2} \quad \text{—(i)}$$

for $n = k+1$, we have;

$$\binom{2(k+1)}{k+1} \leq 2^{2(k+1)-2}$$

$$\binom{2k+2}{k+1} \leq 2^{2k}$$

in eq. (i) $\Rightarrow$ $$\frac{(2k)!}{(2k-k)!(k)!} \leq 2^{2k-2}$$

$$\frac{(2k)!}{k!\,k!} \leq 2^{2k-2}$$

Multiply with $\dfrac{2(2k+1)}{(k+1)}$

Exercise # 7.1

Q.17 $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = 2^n, \forall n \geq 1$

Sol:-

Basis step:-

$\binom{1}{0} = 2^1 \quad , n = 1$

$1 = 1$

Inductive step:-

Put $n = k$

$$\binom{k}{0} + \binom{k}{1} + \dots + \binom{k}{k} = 2^k \quad \text{---(i)}$$

Now for $n = k+1$

$$\binom{k+1}{0} + \binom{k+1}{1} + \dots + \binom{k+1}{k+1} = 2^{k+1} \quad \text{---(ii)}$$

Now add $\binom{k+1}{k+1}$ on b/s in (i)

Now multiply "2" on b/s in (i)

$$\therefore \quad 2\binom{k}{0} + 2\binom{k}{1} + 2\binom{k}{2} + \dots + 2\binom{k}{k} = 2^k \cdot 2$$

$$\binom{k}{0}+\left[\binom{k}{0}+\binom{k}{1}\right]+\left[\binom{k}{1}+\binom{k}{2}\right]+\cdots+\left[\binom{k}{k}\binom{k}{k}\right]=2^{k+1}$$

$$\because \binom{n}{r}+\binom{n}{r+1}=\binom{n+1}{r+1}$$

$$\Rightarrow \binom{k}{0}+\binom{k+1}{1}+\binom{k+1}{2}+\cdots\binom{k+1}{k}+\binom{k}{k}=2^{k+1}$$

$$\text{or} \quad \binom{k+1}{0}+\binom{k+1}{1}+\binom{k+1}{2}+\cdots+\binom{k+1}{k+1}=2^{k+1}$$

$$\because \binom{n}{n}=\binom{n+1}{n+1}$$

Hence proved.

Q. 18 $\binom{n}{1}+2\binom{n}{2}+3\binom{n}{3}+\cdots+n\binom{n}{n}=n.2^{n-1}$

$\forall n \geq 1$

Sol:

Basis step:-

$$\binom{1}{1}=(1)2^{1-1} \quad , n=1$$

$1=1$

Inductive step:-

Put $n=k$

$$\binom{k}{1}+2\binom{k}{2}+3\binom{k}{3}+\cdots+k\binom{k}{k}=k.2^{k-1} \quad \text{—(i)}$$

Now put $n = k+1$

$$\therefore \binom{k+1}{1} + 2\binom{k+1}{1} + 3\binom{k+1}{3} + \dots + (k+1)\binom{k+1}{k+1} = (k+1).2^k$$

Now multiply "2" and add 2^k on b|s in (i)

$$\Rightarrow 2\left[\binom{k}{1} + 2\binom{k}{2} + 3\binom{k}{3} + \dots + k\binom{k}{k}\right] + 2^k = 2(k)(2^{k-1}) + 2^k$$

$$2\left[\binom{k}{1} + 2\binom{k}{2} + 3\binom{k}{3} + \dots + \binom{k}{k}\right] + 2^k = 2^k(k+1)$$

$$\because \quad 2^k = \binom{k}{0} + \binom{k}{1} + \binom{k}{2} + \dots + \binom{k}{k}$$

$$2\left[\binom{k}{1} + 2\binom{k}{2} + 3\binom{k}{3} + \dots + k\binom{k}{k}\right] + \binom{k}{0} + \binom{k}{1} + \binom{k}{2} + \dots + \binom{k}{k} = 2^k(k+1)$$

$$\left[\binom{k}{0} + \binom{k}{1}\right] + 2\left[\binom{k}{1} + \binom{k}{2}\right] + 3\left[\binom{k}{2} + \binom{k}{3}\right] + \dots + k\left[\binom{k}{k}\right] + \binom{k}{k} = 2^k(k+1)$$

$$\therefore \binom{k+1}{1} + 2\binom{k+1}{2} + 3\binom{k+1}{3} + \dots + (k+1)\binom{k+1}{k+1} = 2^k(k+1)$$

Hence proved.

Q 19. $\binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \dots + \frac{1}{n+1}\binom{n}{n} = \frac{2^{n+1} - 1}{n+1}$

$\forall\, n \geq 0$

Sol:-

Basis step:-

$\binom{0}{0} = \frac{2^{0+1} - 1}{0+1}$, $n = 0$

$1 = 1$

Inductive step:-

Put $n = k$

$\binom{k}{0} + \frac{1}{2}\binom{k}{1} + \frac{1}{3}\binom{k}{2} + \dots + \frac{1}{k+1}\binom{k}{k} = \frac{2^{k+1} - 1}{k+1}$ — (i)

Now put $n = k+1$.

$\Rightarrow \binom{k+1}{0} + \frac{1}{2}\binom{k+1}{1} + \dots + \frac{1}{k+2}\binom{k+1}{k+1} = \frac{2^{k+2} - 1}{k+2}$

As we know that

$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \dots + \binom{n}{n} = 2^n$

Put $n = k+2$

$\binom{k+2}{0} + \binom{k+2}{1} + \binom{k+2}{2} + \binom{k+2}{3} + \dots + \binom{k+2}{k+2} = 2^{k+2}$

$$1 + \binom{k+2}{1} + \binom{k+2}{2} + \dots + \binom{k+2}{k+2} = 2^{k+2}$$

$$\binom{k+2}{1} + \binom{k+2}{2} + \binom{k+2}{3} + \dots + \binom{k+2}{k+2} = 2^{k+2} - 1$$

$$\frac{(k+2)!}{(k+1)!\,(1)!} + \frac{(k+2)!}{k!\cdot 2!} + \dots + \frac{(k+2)!}{(k+2)!} = 2^{k+2} - 1$$

$$\frac{(k+2)\cancel{(k+1)!}}{\cancel{(k+1)!}} + \frac{(k+2)\cancel{(k+1)!}}{k\,\cancel{(k+1)!}\,2} + \dots + \frac{(k+2)\cancel{(k+1)!}}{(k+2)\cancel{(k+1)!}} = 2^{k}$$

$$(k+2) + \frac{(k+2)}{2k} + \dots + \frac{(k+2)}{(k+2)} = 2^{k+2} - 1$$

$$(k+2)\left[1 + \frac{1}{2k} + \dots + \frac{1}{(k+2)}\right] = 2^{k+2} - 1$$

$$\frac{(k+1)!}{(k+1)!} + \frac{(k+1)!}{2k(k+1)!} + \frac{(k+1)!}{3\cdot 2!\,(k-1)!} + \dots + \frac{1}{(k+2)}\cdot\frac{(k+1)!}{(k+1)!} = \frac{2^{k+2}-1}{k+2}$$

$$\binom{k+1}{0} + \frac{1}{2}\binom{k+1}{1} + \frac{1}{3}\binom{k+1}{2} + \dots \frac{1}{(k+2)}\binom{k+1}{k+1} = \frac{2^{k+2}-1}{2}$$

Hence proved.